Note on field of norms

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1 APF extension

Throughout this talk, we always assume K is a complete discrete valuation field with perfect residue field k of characteristic p. We always fix a separable closure $\bar{K}$ of K. For any separable extension L/K, let $\mathcal{O}_L$ be the ring of integers, k_L the residue field of L, ν_L the normalised valuation on L (if L/K is finite) and G_L the absolute Galois group of L. Let $U_K = \mathcal{O}_K^\times$ and for any $n \geq 1$, $U_K^n = \{x \in U_K \mid \nu_K(x-1) \geq n\}$.

1.1 Quick review on ramification theory

Let us recall some basic facts on ramification theory. A good reference is Serre's book [Se], espectially chatper IV.

Definition 1.1. Let L/K be a finite separable extension and for any $1 \neq \sigma : L \to \bar{K}$ in $\mathrm{Hom}_K(L, \bar{K})$ (where 1 denotes the natural inclusion $L \subset \bar{K}$), define

$$i_L(\sigma) = \min_{x \in \mathcal{O}_L} (\nu_L(\sigma(x) - x) - 1) \ (i_L(1) := +\infty).$$

Equivalently, for any fixed uniformizer π of L,

$$i_L(\sigma) = \begin{cases} \nu_L(\frac{\sigma(\pi)}{\pi} - 1), & \text{if } \sigma \text{ acts on } k_L \text{ trivially} \\ -1, & \text{else} \end{cases}.$$

Lemma 1.2 (*[Se, p63,Prop 3]*)**.** *Let L'/K be a finite separable extension of L/K. Then*

$$i_L(\sigma) + 1 = \frac{1}{e_{L'/L}} \sum_{\sigma' \mapsto \sigma} (i_{L'}(\sigma') + 1),$$

where σ' runs over the subset of liftings of σ in $\mathrm{Hom}_K(L', \bar{K})$.

A basic tool to study ramification theory is Herbrand's ϕ-function (and ψ-function).

Definition 1.3. Let L/K be a finite separable extension. For any $t \geq -1$, put

$$\gamma_t := \sharp\{\sigma \in \mathrm{Hom}_K(L, \bar{K}) \mid i_L(\sigma) \geq t\}.$$

Then **Herbrand's ϕ-function** is defined as

$$\phi_{L/K}(u) = \begin{cases} u, & -1 \leq u \leq 0 \\ \int_0^u \frac{\gamma_t}{\gamma_0} dt, & u \geq 0 \end{cases}.$$

This is a strictly increasing function and we define **Herbrand's ψ-function** by $\psi_{L/K} = \phi_{L/K}^{-1}$.

Lemma 1.4 (*[Se, p74, Prop 15, Lem 4]*)**.** *Let $K \subset L \subset L'$ be finite separable extensions. Then*

(1) $\phi_{L'/K} = \phi_{L/K} \circ \phi_{L'/L}$ *and* $\psi_{L'/K} = \psi_{L'/L} \circ \psi_{L/K}$.

(2) For any $\sigma \in \mathrm{Hom}_K(L, \bar{K})$, let $j(\sigma) = \sup_{\sigma' \mapsto \sigma} i_{L'}(\sigma')$, then $i_L(\sigma) = \phi_{L'/L}(j(\sigma))$.

Definition 1.5. Let L/K be a finite Galois extension. For any $u \geq -1$, define $\mathrm{Gal}(L/K)_u := \{\sigma \in \mathrm{Gal}(L/K) \mid i_L(\sigma) \geq u\}$ and $\mathrm{Gal}(L/K)^u := \mathrm{Gal}(L/K)_{\psi_{L/K}(u)}$. Define $G_K^u = \varprojlim_{L/K \text{ finite Galois}} \mathrm{Gal}(L/K)^u$.

Lemma 1.6 *([Se, p74, Prop 14]). Let L/K be a finite Galois extension and F/K be a subextension. Then for any $u \geq -1$,*

(1) $\mathrm{Gal}(L/F)^{\psi_{F/K}(u)} = \mathrm{Gal}(L/F) \cap \mathrm{Gal}(L/K)^u$;

(2) If moreover F/K is Galois, then $\mathrm{Gal}(F/K)^u = \mathrm{Gal}(L/K)^u\mathrm{Gal}(L/F)/\mathrm{Gal}(L/F)$.

Remark 1.1. The function $u \mapsto G_K^u$ is semi-continuous: For any $u \geq -1$, $G_K^{<u} := \cap_{v<u} G_K^v = G_K^u$. However, $G_K^{>u} := \cup_{v>u} G_K^v$ may be not G_K^u. For example, $G_K^0 = \mathrm{Gal}(\bar{K}/K^{ur})$ while $G_K^{>0} = \mathrm{Gal}(\bar{K}/K^{tame})$.

1.2 APF extension

Definition 1.7. An extension L/K in $\bar{K}$ is called **arithmetic profinite (APF)** if for any $u \geq -1$, the group $G_K^u G_L$ is open in G_K. In this case, we define $i(L/K) := \sup\{i \geq -1 \mid G_K^i G_L = G_K\}$. For any APF extension L/K, we can also define Herbrand ψ-function by

$$\psi_{L/K}(u) = \begin{cases} u, & -1 \leq u \leq 0 \\ \int_0^u [G_K^0 : G_L^0 G_K^t] dt, & u \geq 0 \end{cases}.$$

An APF extension is called **strictly APF (SAPF)** if

$$\liminf_{u \to +\infty} \frac{\psi_{L/K}(u)}{[G_K^0 : G_L^0 G_K^u]} > 0.$$

When $i(L/K) > 0$, we define

$$c(L/K) = \inf_{u \geq i(L/K)} \frac{\psi_{L/K}(u)}{[G_K^0 : G_L^0 G_K^u]}.$$

Lemma 1.8. *(1) Let L/K be a finite separable extension. Then for any $\sigma \in G_K^u$, we have $i_L(\sigma) \geq \psi_{L/K}(u)$.*

(2) Let L/K be a finite separable extension. Then for any $\sigma \in G_K$, we have $i_L(\sigma) \geq \psi_{L/K}(i(L/K))$.

Proof. For (1): Let L'/K be the Galois closure of L/K. Then $\sigma \in \mathrm{Gal}(L'/K)^u = \mathrm{Gal}(L'/K)_{\psi_{L'/K}(u)}$. Define $j(\sigma) = \sup\{i_{L'}(\tau) \mid \tau \in \mathrm{Gal}(L'/K), \tau_{|L} = \sigma_{|L}\}$. Then by Lemma 1.4,

$$i_L(\sigma) \geq \phi_{L'/L}(j(\sigma)) \geq \phi_{L'/L}(i_{L'}(\sigma)) \geq \phi_{L'/L}(\psi_{L'/K}(u)) = \psi_{L/K}(u).$$

For (2): Since $G_K^{i(L/K)} G_L = G_K$, one can find a $\tau \in G_K^{i(L/K)}$ such that $\tau_{|L} = \sigma_{|L}$. By (1), we have

$$i_L(\sigma) = i_L(\tau) \geq \psi_{L/K}(i(L/K)).$$

□

Example 1.9. (1) Any finite separable extension L/K is (S)APF.

(2) Let L/K be a separable extension with K_0 (resp. K_1) the maximal unramified (resp. tamely ramified) subextenion of K in L. Then L/K is (S)APF if and only if K_i/K is finite (i.e. $G_K^0 G_L$ ($G_K^{>0} G_L$) is open) and L/K_i is (S)APF.

(3) If L/K is APF with $i(L/K) > 0$, then it is SAPF if and only if $c(L/K) > 0$.

Example 1.10 (A conjecture of Serre, confirmed by Sen). Let L/K be a totally ramified Galois extension with $\mathrm{Gal}(L/K)$ a p-adic Lie group (e.g. Lubin–Tate extension). Then L/K is SAPF.

Proposition 1.11. *Let $K \subset L \subset M$ be separable extensions.*

(1) If L/K is finite, then M/K is (S)APF if and only if M/L is.

(2) If M/L is finite, then L/K is (S)APF if and only if M/K is.

(3) If M/K is (S)APF, then so is L/K.

(4) If M/K is APF, then $i(L/K) \geq i(M/K)$. If moreover L/K is finite, then $i(M/L) \geq \psi_{L/K}(i(M/K))$.

(5) If M/K is APF and $i(M/K) > 0$, then $c(L/K) \geq c(M/K)$. If moreover L/K is finite, then $c(M/L) \geq c(M/K)$.

Proof. We only prove (3)-(5) here while the (1) and (2) are easy to believe in.

The APF part of (3) follows from that $[G_K : G_K^u G_L] \leq [G_K : G_K^u G_M]$ and SAPF part will follow from (5) together with Example 1.9 (3).

For (4): Put $i_0 = i(M/K)$. Since

$$G_K = G_M G_K^{i_0} \subset G_L G_K^{i_0} \subset G_K,$$

we have $i(L/K) \geq i_0$. Now assume moreover L/K is finite, then by Lemma 1.6 (1), we have $G_L^{\psi_{L/K}(u)} = G_L \cap G_K^u$. So we get

$$G_L^{\psi_{L/K}(i_0)} G_M = (G_L \cap G_K^{i_0}) G_M = G_L \cap G_K^{i_0} G_M = G_M;$$

So $i(M/L) \geq \psi_{L/K}(i_0)$.

For (5): Note that for any $t \geq 0$, we have

$$[G_K^0 : G_K^t G_M^0] = [G_K^0 : G_K^t G_L^0][G_K^t G_L^0 : G_K^t G_M^0] = [G_K^0 : G_K^t G_L^0][G_L^0 : (G_K^t \cap G_L^0) G_M^0].$$

So we get

$$\begin{aligned}\psi_{M/K}(u) = \int_0^u [G_K^0 : G_K^t G_M^0] dt &\leq \int_0^u [G_K^0 : G_K^t G_L^0] dt \cdot [G_L^0 : (G_K^u \cap G_L^0) G_M^0] \\ &= \frac{[G_K^0 : G_K^u G_M^0]}{[G_K^0 : G_K^u G_L^0]} \int_0^u [G_K^0 : G_K^t G_L^0] dt.\end{aligned}$$

In other words, $\frac{\int_0^u [G_K^0:G_K^tG_M^0]dt}{[G_K^0:G_K^uG_M^0]} \le \frac{\int_0^u [G_K^0:G_K^tG_L^0]dt}{[G_K^0:G_K^uG_L^0]}$. Since $i(L/K) \ge i(M/K)$, we get

$$c(M/K) = \inf_{u\ge i(M/K)} \frac{\int_0^u [G_K^0 : G_K^t G_M^0]dt}{[G_K^0 : G_K^u G_M^0]} \le \inf_{u\ge i(L/K)} \frac{\int_0^u [G_K^0 : G_K^t G_L^0]dt}{[G_K^0 : G_K^u G_L^0]} = c(L/K).$$

If moreover L/K is finite, then

$$[G_K^0 : G_K^t G_M^0] = [G_K^0 : G_K^t G_L^0][G_L^0 : (G_K^t \cap G_L^0)G_M^0] = [G_K^0 : G_K^t G_L^0][G_L^0 : G_L^{\psi_{L/K}(t)} G_M^0].$$

So $[G_K^0 : G_K^u G_M^0] \ge [G_L^0 : G_L^{\psi_{L/K}(u)} G_M^0]$. Since $\psi_{M/K}(u) = \psi_{M/L}(\psi_{L/K}(u))$, we get

$$\frac{\psi_{M/K}(u)}{[G_K^0 : G_K^u G_M^0]} \le \frac{\psi_{M/L}(\psi_{L/K}(u))}{[G_L^0 : G_L^{\psi_{L/K}(u)} G_M^0]}.$$

Since $i(M/L) \ge \psi_{L/K}(i(M/K))$, we get

$$c(M/K) = \inf_{u\ge i(M/K)} \frac{\psi_{M/K}(u)}{[G_K^0 : G_K^u G_M^0]} \le \inf_{v\ge \psi_{L/K}(i(M/K))} \frac{\psi_{M/L}(v)}{[G_L^0 : G_L^v G_M^0]} \le \inf_{v\ge i(M/L)} \frac{\psi_{M/L}(v)}{[G_L^0 : G_L^v G_M^0]} = c(M/L).$$

□

1.3 Elementary extension

Definition 1.12. Let $i > 0$ be a rational number. An finite separable extension L/K is called **elementary of level** i, if $G_K^i G_L = G_K$ and $G_K^{>i} G_L = G_L$. In this case, L/K is totally widely ramified with degree $[L : K]$ a power of p and Herbrand ψ-function

$$\psi_{L/K}(u) = \begin{cases} u, & -1 \le u \le i \\ i + [L : K](u - i), & u \ge i \end{cases}.$$

Let L/K be an infinite APF extension and $B := \{b > 0 \mid G_K^b G_L \ne G_K^{>b} G_L\}$. Then B is infinite (as $[L : K] = +\infty$) and for any $x \ge 0$, $B \cap [-1, x]$ is a finite set (as L/K is APF). So we may write

$$B = \{b_1 \le b_2 \le \cdots\}.$$

For any $n \ge 1$, let $K_n = (\bar{K})^{G_K^{b_n} G_L}$ and $i_n = \psi_{L/K}(b_n)$. Let K_0 be the maximal unramified subextension of K in L. Then we have

(1) For any $n \ge 0$, K_n/K is finite and $L = \cup_{n\ge 0} K_n$.

(2) K_1/K is the maximal tamely ramified subextension of K in L.

(3) For any $n \ge 1$, K_{n+1}/K_n is an elementary extension of level i_n.

(4) $c(L/K_1) = \inf_{n\ge 1} \frac{i_n}{[K_{n+1}:K]}$.

We call $K_0 \subset K_1 \subset \cdots$ the **elementary chain** of infinite APF extension L/K.

Conversely, let $K_0 \subset K_1 \subset \cdots$ be a chain of finite separable extensions of K such that

(1) K_0/K is unramified and K_1/K_0 is totally tamely ramified;

(2) For any $n \geq 1$, K_{n+1}/K_n is an elementary extension of level $i_n > 0$

(3) $L := \cup_{n\geq 0} K_n$. Put $i_0 = 0$ and for any $n \geq 1$, define

$$b_n := \sum_{m=1} \frac{i_m - i_{m-1}}{[K_m : K_0]}.$$

Then L/K is an infinite APF extension if and only if $\lim_{n\to+\infty} b_n = +\infty$ and in this case, $K_0 \subset K_1 \subset \cdots$ is the elementary chain of L/K.

Remark 1.2. The above construction also works for a finite extension L/K. In this case, the set B is finite and hence the elementary chain of L/K is also finite.

1.4 A typical example: Lubin–Tate extension

Now, let K be a local field with residue field $k_K \cong \mathbb{F}_q$ and π be a fixed uniformizer. Fix a polynomial $f(T) = T^q + \cdots + \pi T \in T^q + \pi T\mathcal{O}_K[T]$. Then f determines a unique formal group law $[+]_f$ on $\mathfrak{P}_{\bar{K}}$ such that $[\pi]_f(T) = f(T)$. For any $m \geq 0$, define $\Lambda_{f,m} = \mathrm{Ker}([\pi^{m+1}]_f)$, which is a finite free $\mathcal{O}_K/\pi^{m+1}$-module of rank 1. Let $L_{f,m} = K(\Lambda_{f,m})$ and $L_{f,\infty} = \cup_{m\geq 0} L_{f,m}$. Then Lubin–Tate theory tells us that for any $0 \leq m \leq \infty$, $L_{f,m}/K$ is a Galois extension with Galois group $\mathrm{Gal}(L_{f,m}/K) \cong U_K/U_K^{m+1}$. More precisely, the above isomorphism is induced by a Lubin–Tate character $\chi : \mathrm{Gal}(L_{f,\infty}/K) \to U_K$ such that for any $\lambda \in \Lambda_{f,m}$ and $\sigma \in \mathrm{Gal}(L_{f,\infty}/K)$,

$$\sigma(\lambda) = [\chi(\sigma)]_f(\lambda).$$

Let λ_m be an $\mathcal{O}_K/\pi^{m+1}$-basis of $\Lambda_{f,m}$, which turns out to be a uniformizer of $L_{f,m}$. Then for any $-1 \leq n \leq m$, $\sigma \in \mathrm{Gal}(L_{f,m}/L_{f,n}) \setminus \mathrm{Gal}(L_{f,m}/L_{f,n+1})$ if and only if there exists a basis λ'_{m-n-1} of $\Lambda_{f,m-n-1}$ such that

$$\sigma(\lambda_m) = \lambda_m [+]_f \lambda'_{m-n-1}.$$

Since $X[+]_f Y \equiv X + Y \mod XY$, for such a σ, we have

$$\nu_{L_{f,m}}(\sigma(\lambda_m) - \lambda_m) = \nu_{L_{f,m}}(\lambda'_{m-n-1}) = q^{n+1}.$$

So $i_{L_{f,m}}(\sigma) = q^{n+1} - 1$ if and only if $\sigma \in \mathrm{Gal}(L_{f,m}/L_{f,n}) \setminus \mathrm{Gal}(L_{f,m}/L_{f,n+1})$.

From this, it is easy to see that

$$\mathrm{Gal}(L_{f,m}/K)_u = \begin{cases} \mathrm{Gal}(L_{f,m}/K), & -1 \leq u \leq 0 \\ \mathrm{Gal}(L_{f,m}/L_{f,i}), & q^i - 1 < u \leq q^{i+1} - 1 \ (\forall\ 0 \leq i \leq m-1) \\ 1, & u > q^m - 1 \end{cases}. \tag{1.1}$$

It is easy to compute Herbrand's ψ-function

$$\psi_{L_{f,m}/K}(u)=\begin{cases} u, & -1\le u\le 0 \\ q^i-1+(q^{i+1}-q^i)(u-i), & i<u\le i+1\ (\forall\ 0\le i\le m-1) \\ q^m-1+(q^{m+1}-q^m)(u-m), & u\ge m \end{cases}, \tag{1.2}$$

and ramification groups

$$\mathrm{Gal}(L_{f,m}/K)^u=\begin{cases} \mathrm{Gal}(L_{f,m}/K), & -1\le u\le 0 \\ \mathrm{Gal}(L_{f,m}/L_{f,i}), & i<u\le i+1\ (\forall\ 0\le i\le m-1) \\ 1, & u>m \end{cases}. \tag{1.3}$$

By letting $m\to+\infty$, we conclude that

$$\psi_{L_{f,\infty}/K}(u)=\begin{cases} u, & -1\le u\le 0 \\ q^i-1+(q^{i+1}-q^i)(u-i), & i<u\le i+1\ (\forall\ 0\le i) \end{cases}, \tag{1.4}$$

and that

$$\mathrm{Gal}(L_{f,\infty}/K)^u=\begin{cases} \mathrm{Gal}(L_{f,\infty}/K), & -1\le u\le 0 \\ \mathrm{Gal}(L_{f,\infty}/L_{f,i}), & i<u\le i+1\ (\forall\ 0\le i) \end{cases}. \tag{1.5}$$

From this, we see that

Proposition 1.13. *Keep notations as above.*

(1) $G_K^uG_{L_{f,\infty}}\neq G_K^{>u}G_{L_{F,\infty}}$ *if and only if* $u\in\mathbb{N}_{\ge 0}$. *In particular,* $i(L_{f,\infty}/K)=0$.

(2) $L_{f,0}/K$ *is a totally ramified extension of degree* $q-1$.

(3) For any $m\ge 0$, $L_{f,m+1}/L_{f,m}$ *is an elementary extension of level* $q^{m+1}-1$.

(4) $i(L_{f,\infty}/L_{f,0})=q-1$ *and* $c(L_{f,\infty}/L_{f,0})=1-\frac{1}{q}$. *In particular,* $L_{f,\infty}/K$ *is SAPF.*

(5) $K=K_0\subset L_{f,0}=K_1\subset L_{f,1}=K_2\subset\cdots$ *is the elementary chain of* $L_{f,\infty}/K$.

Remark 1.3. Recall that Hasse–Arf theorem says that for any finite abelian extension L/K of local fields, the jumps of the function $u\mapsto\mathrm{Gal}(L/K)^u$ are all integers. Lubin–Tate theory tells us that the maximal abelian extension $K^{ab}=K^{ur}L_{f,\infty}$. So one can recover Hasse–Arf theorem from the above proposition.

2 The field of norms

From now on, we assume L/K is an infinite APF extension and define

$$\mathcal{E}_{L/K}:=\{E\mid K\subset E\subset L,[E:K]<+\infty\}.$$

Clearly, $\mathcal{E}_{L/K}$ is a filtered category.

2.1 The construction of X_K

Definition 2.1. Define $X_K(L) := \varprojlim_{E\in\mathcal{E}_{L/K}} E$, where the translation maps are norm maps. We denote by $\underline{x} = (x_E)_E$ the elements of $X_K(L)$.

It is easy to see that $X_K(L)$ is a commutative monoid.

Remark 2.1. Let $\mathcal{E} \subset \mathcal{E}_{L/K}$ be a cofinal subset. Then we have $X_K(L) = \varprojlim_{E\in\mathcal{E}} E$.

Construction 2.2. *For any $a \in k_L$, let $[a]$ be its Teichimüller lifting in K_0. For any $E \in \mathcal{E}_{L/K_1}$, $[a^{\frac{1}{[E:K_1]}}]$ is a well-defined element in E such that $f_{L/K}(a) := ([a^{\frac{1}{[E:K_1]}}])_{E\in\mathcal{E}_{L/K_1}}$ is a well-defined element in $X_K(L)$. So we get a morphism of monoids $f_{L/K} : k_L \to X_K(L)$. For any $\underline{x} \in X_K(L)$, the value $\nu_E(x_E)$ is independent of the choice of $E \in \mathcal{E}_{L/K_1}$ and we denote this value by $\nu(\underline{x})$. Let $\mathcal{O}_{X_K(L)} = \{\underline{x} \in X_K(L) \mid \nu(\underline{x}) \geq 0\}$.*

A key ingredient is the following proposition:

Proposition 2.3. *Let $\underline{x}, \underline{y} \in X_K(L)$. Then for any $E \in \mathcal{E}_{L/K_1}$, $\{N_{F/E}(x_F + y_F)\}_{F\in\mathcal{E}_{L/E}}$ converges to a unique element $z_E \in E$.*

It is easy to check that $\underline{z} = (z_E)_E$ is a well-defined element in $X_K(L)$. We define $\underline{x} + \underline{y} := \underline{z}$.

Corollary 2.4. *$X_K(L)$ is a field under addition defined above.*

Proof. It is easy to check $X_K(L)$ is a ring and then the corollary follows from that

$$X_K(L) \setminus \{0\} = \varprojlim_{E\in\mathcal{E}_{L/K}} E^\times$$

is a group. □

The main result is

Theorem 2.5. *The $X_K(L)$ is a complete discrete valuation field of characteristic p and ν is the normalised valuation on $X_K(L)$. The map $f_{L/K} : k_L \to X_K(L)$ identifies k_L with the residue field of $X_K(L)$.*

Remark 2.2. The field $X_K(L)$ is called the **field of norms** with respect to the APF extension L/K.

Example 2.6 (Lubin–Tate case). Let $L_{f,\infty}/K$ be the Lubin–Tate extension that we studied in the previous section. Then $X_K(L_{f,\infty}) = \varprojlim_n L_{f,n}$. Let λ_m be the basis of $\Lambda_{f,m}$ such that $[\pi]_f(\lambda_{m+1}) = \lambda_m$. Then we have $N_{L_{f,m+1}/L_{f,m}}(\lambda_{m+1}) = \lambda_m$. In particular, $\underline{\lambda} := (\lambda_m)_{m\geq 0}$ defines an element of $X_K(L_{f,\infty})$, which is obviously a uniformizer. Therefore, we see that $X_K(L_{f,\infty}) \cong k_K((\underline{\lambda}))$. For example, if $K = \mathbb{Q}_p$, $f(T) = (1+T)^p - 1$ and $L_{f,m} = \mathbb{Q}_p(\zeta_{p^{m+1}})$, then we have $X_{\mathbb{Q}_p}(\mathbb{Q}_p(\zeta_{p^\infty})) = \mathbb{F}_p((X))$, where $X = (\zeta_{p^{m+1}} - 1)_{m\geq 0}$.

2.2 Some preparations

We need some preparations to prove Theorem 2.5.

Proposition 2.7. *Let E/K be a totally ramified finite separable extension of degree p^r. Then for any $x, y \in \mathcal{O}_E$, we have*

$$\nu_K(N_{E/K}(x+y) - N_{E/K}(x) - N_{E/K}(y)) \geq \frac{p-1}{p} i(E/K).$$

An immediate corollary is

Corollary 2.8. *For any $a \in \mathcal{O}_K$, there exists an $x \in \mathcal{O}_E$ such that $\nu_K(N_{E/K}(x) - a) \geq \frac{p-1}{p} i(E/K)$.*

Proof. Let π_E be a uniformizer of E. Then $\pi_K := N_{E/K}(\pi_E)$ is a uniformizer of K. For any $a \in \mathcal{O}_K$, it is of the form $a = \sum_{n\geq 0}[a_n]\pi_K^n$ with $a_n \in k_K$. Then one can check that $x = \sum_{n\geq 0}[a_n^{\frac{1}{p^r}}]\pi_E^n$ works. ☐

Proof of Proposition 2.7. **Step 1:** We first show that if F/K is a subextension in E such that the result holds for E/F and F/K, then the result is true for E/K.

Indeed, for any $x, y \in \mathcal{O}_E$, by Proposition 2.7 for E/F, there exists a $z \in \mathcal{O}_F$ with $\nu_F(z) \geq \frac{p-1}{p} i(E/F)$ such that

$$N_{E/F}(x+y) = N_{E/F}(x) + N_{E/F}(y) + z.$$

By Proposition 2.7 for F/E, there exists an $a \in \mathcal{O}_K$ with $\nu_K(a) \geq \frac{p-1}{p} i(F/K)$ such that

$$N_{F/K}(N_{E/F}(x) + N_{E/F}(y) + z) = N_{E/K}(x) + N_{E/K}(y) + N_{F/K}(z) + a.$$

So we have

$$\begin{aligned}
\nu_K(N_{E/K}(x+y) - N_{E/K}(x) - N_{E/K}(y)) &= \nu_K(N_{F/K}(z) + a) \\
&\geq \min(\nu_K(N_{F/K}(z)), \nu_K(a)) \\
&\geq \frac{p-1}{p} \min(i(E/F), i(F/K)) \\
&\geq \frac{p-1}{p} i(E/K) \quad \text{(cf. Prop 1.11(4))}
\end{aligned}$$

Step 2: We show the result is true when E/K is Galois. Since $\mathrm{Gal}(E/K)$ is a p-group (and hence solvable), by Step 1, we may assume E/K is moreover cyclic of degree p.

We may assume $\nu_E(x) \geq \nu_E(y)$ such that $y \neq 0$. Replacing x and y by $\frac{x}{y}$ and 1, we may assume $y = 1$. By the following lemma:

Lemma 2.9 (*[Se, p83, Lem 5]*). *Let E/K be a totally ramified cyclic extension of degree p. Then for any $n \geq 0$ and any $x \in \mathcal{O}_E$ with $\nu_E(x) \geq n$, we have*

$$N_{E/K}(1+x) \equiv 1 + N_{E/K}(x) + T_{E/K}(x) \mod T_{E/K}(\mathfrak{P}_E^{2n}).$$

we see that $N_{E/K}(1+x) - 1 - N_{E/K}(x) \in T_{E/K}(\mathcal{O}_E)$. By the following lemma:

Lemma 2.10 (*[Se, p83, Lem 4]*). *Let E/K be a totally ramified cyclic extension of degree p and $m := (i(E/K)+1)(p-1)$. Then for any $n \geq 0$,*

$$T_{E/K}(\mathfrak{P}_E^n) = \mathfrak{P}_K^{[\frac{m+n}{p}]}.$$

We see that

$$\nu_K(N_{E/K}(1+x) - 1 - N_{E/K}(x)) \geq [\frac{(i(E/K)+1)(p-1)}{p}] \geq \frac{p-1}{p} i(E/K)$$

as desired. Here, we apply Hasse–Arf theorem (i.e. $i(E/K) \in \mathbb{N}$) implicitly.

Step 3: Assume E/K is a subextension of some totally ramified Galois extension F/K of degree p^n. Then the result holds true for E/K.

Indeed, $\mathrm{Gal}(F/E)$ is a subgroup of the p-group $\mathrm{Gal}(F/K)$. Use the following well-known lemma:

Lemma 2.11. *Let G be a p-group and $H < G$ be a subgroup. Then $H < N_G(H)$ is a strict subgroup of its normalizer in G.*

By Galois correspondence, we know that F/K factors as suquential Galois extensions (which are totally widely ramified). So we conclude by first two steps.

Step 4: Now let F be the Galois closure of E/K and K_1 be the maximal tamely ramified subextension of K in F. Then E and F are linearly disjoint over K. In particular, we have

$$N_{E/K}(x+y) - N_{E/K}(x) - N_{E/K}(y) = N_{EK_1/K_1}(x+y) - N_{EK_1/K_1}(x) - N_{EK_1/K_1}(y).$$

By Step 3, we have

$$\nu_{K_1}(N_{E/K}(x+y) - N_{E/K}(x) - N_{E/K}(y)) \geq \frac{p-1}{p} i(EK_1/K_1).$$

Then the result follows from that $\nu_{K_1} = e_{K_1/K}\nu_K$ and that

Lemma 2.12. $i(EK_1/K_1) = e_{K_1/K} i(E/K)$.

Proof. Recall if M/N is a tamely ramified extension, then we have $\psi_{M/N}(u) = e_{M/N}u$ when $u \geq 0$. Since $\psi_{EK_1/K} = \psi_{EK_1/K_1} \circ \psi_{K_1/K} = \psi_{EK_1/E} \circ \psi_{E/K}$, the result follows by comparing the first cusp of $\psi_{EK_1/K}(u)$ $(u > 0)$. □

Now, the proof is complete. □

Proposition 2.13. *Let E/K be a totally ramified separable extension of degree p^r. Then for any $x, y \in \mathcal{O}_E$ such that $\nu_E(x-y) \geq n$, we have*

$$\nu_K(N_{E/K}(x) - N_{E/K}(y)) \geq \phi_{E/K}(n).$$

Proof. As the proof of Proposition 2.7, we may assume E/K is a moreover a Galois extension. We may assume $\nu_E(x) \geq \nu_E(y)$ and $y \neq 0$. Noting that

$$\nu_K(N_{E/K}(\frac{x}{y}) - 1) = \nu_K(N_{E/K}(x) - N_{E/K}(y)) - \nu_E(y)$$

and that

$$\phi_{E/K}(n - \nu_E(y)) \geq \phi_{E/K}(n) - \nu_E(y),$$

we may assume $y = 1$. When $n = 0$, the result is trivial. So we may assume $n \geq 1$; equivalently, $x \in U_E^n$ and are reduced to showing that $\nu_K(N_{E/K}(x) - 1) \geq \phi_{E/K}(n)$.

Lemma 2.14 (*[Se, p91, Prop 8]*)**.** *Let E/K be a totally ramified Galois extension, then for any $m \geq 0$, we have $N_{E/K}(U_E^{\psi_{E/K}(m)}) \subset U_K^m$ and $N_{E/K}(U_E^{\psi_{E/K}(m)+1}) \subset U_K^{m+1}$.*

Let m be the integer satisfying $\psi_{E/K}(m) \leq n < \psi_{E/K}(m+1)$. If $\psi_{E/K}(m) = n$, by above lemma, we have $\nu_K(N_{E/K}(x) - 1) \geq m = \phi_{E/K}(n)$. If $\psi_{E/K}(m) < n$, we have $\nu_K(N_{E/K}(x) - 1) \geq m + 1 \geq \phi_{E/K}(n)$, again by above lemma. The proof is complete. □

Now, we are able to prove Proposition 2.3.

Proof of Proposition 2.3: Let $\underline{x}, \underline{y} \in X_K(L)$. Fix an $E \in \mathcal{E}_{L/K_1}$.

Let $F_1 \subset F_2$ be elements in $\mathcal{E}_{L/E}$. Then by Proposition 2.7, we have

$$\nu_{F_1}(x_{F_1} + y_{F_1} - N_{F_2/F_1}(x_{F_2} + y_{F_2})) \geq \frac{p-1}{p} i(F_2/F_1) \geq \frac{p-1}{p} i(L/F_1).$$

By Proposition 2.13, we have

$$\nu_E(N_{F_1/E}(x_{F_1} + y_{F_2}) - N_{F_2/E}(x_{F_2} + y_{F_2})) \geq \phi_{F_1/E}(\frac{p-1}{p} i(L/F_1)) \geq \phi_{L/E}(\frac{p-1}{p} i(L/F_1)).$$

It remains to show $\lim_{F \to L} i(L/F) = +\infty$: Let $K_0 \subset K_1 \subset \cdots$ be the elementary chain of L/K and then we have $\lim_{n \to +\infty} i(L/K_n) = +\infty$. □

2.3 The proof of main theorem

For any $E \in \mathcal{E}_{L/K}$, define $r(E) := \min\{n \in \mathbb{N} \mid n \geq \frac{p-1}{p} i(L/E)\}$. We have shown that $\lim_{E \to L} r(E) = +\infty$ and if $E_1 \subset E_2$, then $r(E_1) \leq r(E_2)$ (cf. Proposition 1.11 (4)).

Construction 2.15. *For any $E \in \mathcal{E}_{L/K_1}$, define $\bar{A}_E := \mathcal{O}_E/\mathfrak{P}_E^{r(E)}$. By Proposition 2.7 and Corollary 2.8, for any $F \in \mathcal{E}_{L/E}$, the norm map $N_{F/E} : \bar{A}_F \to \bar{A}_E$ is a surjective homomorphism of rings. Define*

$$A_K(L) := \varprojlim_{E \in \mathcal{E}_{L/K_1}} \bar{A}_E.$$

Then $A_K(L)$ is a ring.

Let $0 \neq \underline{x} = (\bar{x}_E)_E \in A_K(L)$. Assume $\bar{x}_E \neq 0$ and x_E is a lifting of $\bar{x}_E$ in $\mathcal{O}_E$. Then $\nu_E(x_E)$ only depends on $\underline{x}$ and we denote this value by $\nu(\underline{x})$. Obviously, $(A_K(L), \nu)$ is a complete discrete valuation ring, whose residue field is $\varprojlim_{E \in \mathcal{E}_{L/K_1}} k_E \cong k_L$.

There exists a natural morphism $\iota : \mathcal{O}_{X_K(L)} \to A_K(L)$ of monoids by sending $(\underline{x}) = (x_E)_E$ to $\iota(\underline{x}) = (\bar{x}_E)_E$, which clearly preserves ν. In particular, $f_{L/K} : k_L \to X_K(L)$ induces a morphism $k_L \to A_K(L)$ of monoids and an isomorphism of fields $k_L \cong k_{A_K(L)}$.

Lemma 2.16. *The morphism $\iota : \mathcal{O}_{X_K(L)} \to A_K(L)$ is an isomorphism of rings.*

Proof. Since ι preserves ν, it is automatically injective as long as we show it is a ring homomorphism. For this purpose, we need to show ι also preserves additions on both sides. Let $\underline{x}, \underline{y} \in \mathcal{O}_{X_K(L)}$ and $\underline{z} = \underline{x} + \underline{y}$. By Proposition 2.3, for any $E \in \mathcal{E}_{L/K_1}$, we have

$$z_E = \lim_{F \to L} N_{F/E}(x_F + y_F).$$

Taking reduction modulo $\mathfrak{P}_E^{r(E)}$, we see that for F sufficiently close to L,

$$\bar{z}_E = N_{F/E}(\bar{x}_F + \bar{y}_F) = \bar{x}_E + \bar{y}_E,$$

which is exactly what we want. It remains to show ι is surjective. For any $\underline{x} \in A_K(L)$, we choose a lifting $\hat{x}_E$ of $\bar{x}_E$ in $\mathcal{O}_E$. Then for any $F_1 \subset F_2 \in \mathcal{E}_{L/K_1}$, $\nu_{F_1}(N_{F_2/F_1}(\hat{x}_{F_2}) - \hat{x}_{F_1}) \geq r(F_1)$. By Proposition 2.13, we have

$$\nu_E(N_{F_2/E}(\hat{x}_{F_2}) - N_{F_1/E}(\hat{x}_{F_1})) \geq \phi_{F_1/E}(r(F_1)) \geq \phi_{L/E}(r(F_1)).$$

Since $\lim_{F \to L} r(F) = +\infty$, we know that $\{N_{F/E}(\hat{x}_F)\}_{F \in \mathcal{E}_{L/E}}$ converges to a unique element $x_E \in \mathcal{O}_E$ lifting $\bar{x}_E$ and satisfying $N_{F/E}(x_F) = x_E$. So $(x_E)_E \in \mathcal{O}_{X_K(L)}$ which is carried to $\underline{x}$ by ι. □

To conclude Theorem 2.5, we are reduced to the following lemma:

Lemma 2.17. *The map $\iota \circ f_{L/K} : k_L \to A_K(L)$ is an homomorphism of rings.*

Proof. Since for any $a, b \in k_L$, $[a] + [b] \equiv [a+b] \mod p$, it suffices to show that $A_K(L)$ is an $\mathbb{F}_p$-algebra. For this, it is enough to show that for any $E \in \mathcal{E}_{L/K_1}$, $\nu_E(p) \geq \frac{p-1}{p} i(L/E)$. Fix an extension $F \in \mathcal{E}_{L/E}$. We want to show $\nu_E(p) \geq \frac{p-1}{p} i(F/E)$. As in the proof of Proposition 2.7, we are reduced to the case where F/E is a totally ramified finite Galois extension of degree p^r.

Lemma 2.18 ([Se, p71, Exer 3])**.** *Let E/K be a finite Galois extension and $i \geq 1$. If $i \geq \frac{\nu_E(p)}{p-1}$, then* $\mathrm{Gal}(E/K)_i = 1$.

Let E_1/E be subextension in F/E which is totally ramified cyclic of degree p. Then the above lemma implies that

$$i(E_1/E) \leq \frac{\nu_{E_1}(p)}{p-1} = \frac{p}{p-1} \nu_E(p).$$

So we conclude that $\nu_E(p) \geq \frac{p-1}{p} i(E_1/E) \geq \frac{p-1}{p} i(F/E)$. We win! □

3 Functoriality of X_K

In this section, we show X_K is a functor from the category of infinite APF extensions of K to the category of fields in characteristic p. We recall the following result:

Lemma 3.1 *([Se, p89, Lem 6]). Let $L = \cup_{i \in I} L_i$ be an extension of K with I a filtered set and $\{L_i\}_{i \in I}$ an increasing family of subextensions. Let M/L be an extension of degree n. Then there exists an $i \in I$ and an extension M_i/L_i of degree n such that M_i and L are linearly disjoint over L_i and $M_i L = M$. If both M_i and M_j satisfy the above conditions, then there exists a $k \geq i, j$ such that $M_i L_k = M_j L_k = M_k$. In particular, M_k satisfies the same conditions. If moreover M/L is separable (resp. Galois), one may choose M_i such that M_i/L_i is also separable (resp. Galois).*

Remark 3.1. In the case for M/L Galois, we may further assume $\mathrm{Gal}(M_i/L_i) \cong \mathrm{Gal}(M/L)$.

3.1 X_K as a functor

We fix an infinite APF extension L/K.

Construction 3.2. *Let M/K be an infinite APF extension and $\tau : L \to M$ be a K-homomorphism of degree n. We construct a homomorphism $X_K(\tau) : X_K(L) \to X_K(M)$ as follows:*

Put $\mathcal{E}_{M,\tau} = \{F \in \mathcal{E}_{M/K} \mid \tau(L) \otimes_{\tau(L) \cap F} F \cong M\}$ and $\mathcal{E}_{L,\tau} = \{\tau^{-1}(\tau(L) \cap F) \mid F \in \mathcal{E}_{M,\tau}\}$. By Lemma 3.1, both $\mathcal{E}_{L,\tau}$ and $\mathcal{E}_{M,\tau}$ are cofinal in $\mathcal{E}_L$ and $\mathcal{E}_M$, respectively. Then we define

$$X_K(\tau) : X_K(L) = \varprojlim_{E \in \mathcal{E}_{L,\tau}} E \to \varprojlim_{F \in \mathcal{E}_{M,\tau}} F = X_K(M)$$

by sending $\underline{x} = (x_{\tau^{-1}(\tau(L) \cap F)})$ to $(\tau(x_{\tau^{-1}(\tau(L) \cap F)}))_F$. One can check $X_K(\tau)$ is well-defined. Clearly, $X_K(\tau)$ preserves valuations (as τ does so).

Example 3.3. If $\tau : L \to M$ is an isomorphism with inverse τ^{-1}, then $X_K(\tau)$ is also an isomorphism whose inverse is $X_K(\tau^{-1})$.

Proposition 3.4. *The homomorphism $X_K(\tau)$ above is separable of degree n. If moreover $M/\tau(L)$ is Galois, then so is $X_K(M)/X_K(\tau)(X_K(L))$ and in this case, X_K induces an isomorphism*

$$\mathrm{Gal}(M/\tau(L)) \cong \mathrm{Gal}(X_K(M)/X_K(\tau)(X_K(L))).$$

Proof. By Example 3.3, we may assume $\tau : L \to M$ is the natural inclusion $L \subset M$. By Galois correspondence, we may assume M/L is already finite Galois. Let K_0 be the maximal unramified subextension of K in M.

Now, let $\mathcal{E}_{M,G} = \{F \in \mathcal{E}_{M/K_0} \mid L \otimes_{L \cap F} F = M \ \&\ F/L \cap F \text{ is Galois}\}$ and $\mathcal{E}_{L,G} = \{F \cap L \in \mathcal{E}_{L/K_0 \cap L} \mid F \in \mathcal{E}_{M,G}\}$. By Lemma 3.1, both $\mathcal{E}_{L,G}$ and $\mathcal{E}_{M,G}$ are cofinal in $\mathcal{E}_L$ and $\mathcal{E}_M$, respectively. In particular, we have $G = \mathrm{Gal}(M/L) = \mathrm{Gal}(F/F \cap L)$ for any $F \in \mathcal{E}_{M,G}$. By construction of $X_K(\tau)$ above, for any $\sigma \in G$ and any $\underline{x} = (x_F)_{F \in \mathcal{E}_{M,G}} \in X_K(M)$, we have $X_K(\sigma)(\underline{x}) = (\sigma(x_F))_{F \in \mathcal{E}_{M,G}}$. So $X_K(M)^G = X_K(L)$. We claim that G acts on $X_K(M)$ faithfully. Granting this, by Galois' theorem, we see that $X_K(M)/X_K(L)$ is finite Galois with Galois group G.

It remains to check that G acts on $X_K(M)$ faithfully. Let $\sigma \in G$ such that $X_K(\sigma) = 1$. Then we see σ acts trivially on $k_{X_K(M)} \cong k_M$. In particular, for any $F \in \mathcal{E}_{M,G}$ with $E = F \cap L$, we have σ acts on k_F trivially. Let $\underline{\pi} = (\pi_F)_{F \in \mathcal{E}_{M,G}}$ be a uniformizer of $X_K(M)$. Then π_F is also a uniformizer of F for each F. Since $X_K(\sigma)$ acts on $\underline{\pi}$ trivially, we see that $\sigma(\pi_F) = \pi_F$. Therefore, $i_F(\sigma) = +\infty$, which forces that $\sigma = \mathrm{id}_F$. So $\sigma = 1$. □

3.2 Fontaine–Wintenberger's theorem

Construction 3.5. *Let M/L be an algebra separable extension in $\bar{K}$. Then $M = \cup_{E \in \mathcal{E}_{M/L}} E$ and for any $E \in \mathcal{E}_{M/L}$, $X_K(E)$ is well-defined. The functoriality of X_K allows us to define $X_{L/K}(M) := \mathrm{colim}_{E \in \mathcal{E}_{M/L}} X_K(L)$. This is an algebraic separable extension of $X_K(L)$ and if M/L is Galois, then so in $X_{L/K}(M)/X_K(L)$ such that* $\mathrm{Gal}(L/M) \cong \mathrm{Gal}(X_{L/K}(M)/X_K(L))$. *In particular, we can define $X_{L/K}(\bar{K})$.*

The main result is

Theorem 3.6. *The $X_{L/K}(\bar{K})$ is a separable closure of $X_K(L)$. In particular, we have a canonical isomorphism $G_{X_K(L)} \cong G_L$.*

Remark 3.2. When $K = \mathbb{Q}_p$ and $L = \mathbb{Q}_p(\zeta_{p^\infty})$, the isomorphism

$$G_{\mathbb{Q}_p(\zeta_{p^\infty})} \cong G_{\mathbb{F}_p((X))} \cong G_{\mathbb{F}_p((X^{\frac{1}{p^\infty}}))}$$

with $X = (\zeta_{p^{n+1}} - 1)_{n \geq 0}$ is well-known as Fontaine–Wintenberger theorem in classical p-adic Hodge theory.

Theorem 3.6 is an immediate consequence of the following proposition:

Proposition 3.7. *(1) For any separable algebraic extension $X/X_K(L)$, there exists a separable algebraic extension M/L such that $X \cong X_{L/K}(M)$.*

(2) For any separable algebraic extensions M_1 and M_2, we have

$$\mathrm{Hom}_L(M_1, M_2) = \mathrm{Hom}_{X_K(L)}(X_K(M_1), X_K(M_2)).$$

Proof. The item (2) is easy: By several reductions, we may assume M_1 and M_2 are both finite over L. Then by replacing M_2/L by its Galois closure, we may assume M_2/L is finite Galois and then are reduced to Proposition 3.4.

For (1), by functoriality of X_k and Item (2), we may assume $X/X_K(L)$ is finite of degree d.

Let $f(T) = T^d + \underline{a}_1 T^{d-1} + \cdots + \underline{a}_d$ be an irreducible polynomial over $\mathcal{O}_{X_K(L)}$ such that $X \cong X_K(L)[T]/(f(T))$. Let $E_1 \subset E_2 \subset \cdots$ be subextensions in $\mathcal{E}_{L/K_1}$ such that $L = \cup_n E_n$. Then $X_K(L) \cong \varprojlim_n E_n$ and we write $\underline{a}_i = (a_{i,n})_{n\geq 1}$. Define $f_n(T) = T^d + a_{1,n}T^{d-1} + \cdots + a_{d,n}$.

Let $\Delta(g)$ be the discriminant of a polynomial $g(T) = T^d + x_1 T^{d-1} + \cdots + x_d$ over a certain field. Then there exists a polynomial $D(X_1, \ldots, X_d) \in \mathbb{Z}[X_1, \ldots, X_d]$ such that $\Delta(g) = D(x_1, \ldots, x_d)$.

Lemma 3.8. *For $n \gg 0$, $\nu_{X_K(L)}(\Delta(f)) = \nu_{E_n}(\Delta(f_n))$.*

Proof. Recall $\lim_{n\to\infty} r(E_n) = +\infty$. So for $n \gg 0$, $r(E_n) \geq \nu_{X_K(L)}(\Delta(f)) = \nu_{X_K(L)}(D(\underline{a}_1, \ldots, \underline{a}_d))$. Since the coefficients of D belong to $\mathbb{Z}$, we see that

$$\Delta(f) = D(\underline{a}_1, \ldots, \underline{a}_d) = (D(a_{1,n}, \ldots, a_{d,n}))_{n\geq 1} = (\Delta(f_n))_{n\geq 1} \in \varprojlim_n \bar{A}_{E_n} = \mathcal{O}_{E_n}/\mathfrak{P}_{E_n}^{r(E_n)}.$$

So the result follows from the definition of $\nu_{X_K(L)}$. □

In particular, we may assume for any $n \geq 0$, $f_n(T)$ is separable (i.e. $\Delta(f_n) \neq 0$). Let x_n be a root of $f_n(T) = 0$, and let $F_n = E_n(x_n)$ and $L_n = L(x_n) = LF_n$. Since $\lim_{n\to+\infty} i(L/E_n) = +\infty$, we may assume $i(L/E_n) \geq d\nu_{X_K(L)}(\Delta(f))$ for all n. Then we have

Lemma 3.9. *For any $u \geq d\nu_{X_K(L)}(\Delta(f))$, $G_{E_n}^u \subset G_{F_n}$.*

Proof. For any $\sigma \in G_{E_n}^u$, assume $\sigma(x_n) \neq x_n$, we have

$$\begin{aligned} \nu_{F_n}(\sigma(x_n) - x_n) &> \min_{x\in\mathcal{O}_{F_n}} (\nu_{F_n}(\sigma(x) - x) - 1) = i_{F_n}(\sigma) \\ &\geq \psi_{F_n/E_n}(u) \quad \text{(by Lemma 1.8 (1))} \\ &\geq u \geq d\nu_{X_K(L)}(\Delta(f)) = d\nu_{E_n}(\Delta(f_n)) \\ &\geq \nu_{F_n}(\Delta(f_n)) \geq 2\nu_{F_n}(\sigma(x_n) - x_n), \end{aligned}$$

which is impossible. So we must have $\sigma(x_n) = x_n$, which forces that $\sigma \in G_{F_n}$. □

By applying above Lemma to $u = i(L/E_n)$, we see that

$$G_{E_n} = G_{E_n}^{i(L/E_n)} G_L \subset G_{F_n} G_L \subset G_{E_n}.$$

As a consequence, we deduce that L/E_n and F_n/E_n are linearly disjoint:

Lemma 3.10. $E_n = L \cap F_n$.

Using this, one can conclude that

Lemma 3.11. $i(L_n/F_n) = \psi_{F_n/E_n}(i(L/E_n))$.

Proof. Since $G_{E_n}^u \cap G_{F_n} = G_{F_n}^{\psi_{F_n/E_n}(u)}$, by Lemma 3.9, for any $u \geq d\nu_{X_K(L)}(\Delta(f))$, we have

$$G_{E_n}^u = G_{F_n}^{\psi_{F_n/E_n}(u)}.$$

Therefore, for $u \geq i(L/E_n)$, we have

$$G_{F_n}^{\psi_{F_n/E_n}(u)} G_{L_n} = G_{E_n}^u (G_L \cap G_{F_n}) = G_{E_n}^u G_L \cap G_{F_n}.$$

Applying $u = i(L/E_n)$, we have

$$G_{F_n}^{\psi_{F_n/E_n}(i(L/E_n))} G_{L_n} = G_{E_n}^{i(L/E_n)} G_L \cap G_{F_n} = G_{E_n} \cap G_{F_n} = G_{F_n},$$

which implies that $i(L_n/F_n) \geq \psi_{F_n/E_n}(i(L/E_n))$.

If this inequality is strict, then there exists some $j > i(L/E_n)$ such that $G_{F_n}^{\psi_{F_n/E_n}(j)} G_{L_n} = G_{F_n}$, which implies that $G_{F_n} \subset G_{E_n}^j G_L$. Let F be the field such that $G_F = G_{E_n}^j G_L$. Then by the choice of j, we see that F/E_n is a proper extension and $F \subset L$. On the other hand, it follows from that $G_{F_n} \subset G_F$ that $F \subset F_n$. So we see that $F \subset L \cap F_n$ is a proper extension of E_n, which violates to Lemma 3.10. So we deduce $i(L_n/F_n) = \psi_{F_n/E_n}(i(L/E_n))$ as desired. □

In particular, L_n/F_n is totally widely ramified. Let $r_n = \min\{r \in \mathbb{N} \mid r \geq \frac{p-1}{p} i(L_n/F_n)\}$. By Construction 2.15, we see that $\mathcal{O}_{X_K(L_n)} = A_K(L_n) \to \bar{A}_{F_n} = \mathcal{O}_{F_n}/\mathfrak{P}_{F_n}^{r_n}$ is surjective. Let $y_n \in \mathcal{O}_{X_K(L_n)}$ be a lifting of reduction of x_n in $\bar{A}_{F_n}$.

We claim that $\lim_{n\to+\infty} f(y_n) = 0$. Granting this, by replacing $(y_n)_{n\geq 1}$ by a subsequence, we may assume $y = \lim_{n\to+\infty} y_n$ exists. So $f(y) = 0$.

Lemma 3.12 (Krasner's Lemma)**.** *Let K be a complete non-archimedean field with separable closure $\bar{K}$. For any $a \in \bar{K}$ with all conjugations $a_1 = a, a_2, \ldots, a_d$, if $b \in \bar{K}$ such that $|b-a| < \min_{2\leq i\leq d}(|a-a_i|)$, then $K(a) \subset K(b)$.*

For $n \gg 0$, applying Krasner's Lemma to $a = y$ and $b = y_n$, we have

$$X \cong X_K(L)(y) \subset X_K(L)(y_n) \subset X_K(L_n).$$

On the other hand, we have

$$[X : X_K(L)] = d \geq [L_n : L] = [X_K(L_n) : X_K(L)].$$

So we conclude that $X = X_K(L_n)$ and complete the proof.

Now, we are reduced to showing that $\lim_{n\to+\infty} \nu_{X_K(L)}(f(y_n)) = +\infty$. Since L_n/F_n is totally ramified, we have

$$\nu_{X_K(L_n)}(f(y_n)) = \nu_{F_n}(f(y_n)_{F_n}),$$

where $f(y_n)_{F_n}$ is the projection of $f(y_n)$ along $X_K(L_n) \cong \varprojlim_{F \in \mathcal{E}_{L_n/K}} F \to F_n$. Since L/E_n and F_n/E_n are linear disjoint, we see that as an element in $X_K(L) \subset X_K(L_n)$, the projection of $\underline{a}_i$ along $X_K(L_n) \to F_n$ is exactly $a_{i,n}$. By construction of y_n, we know that as an element in $\bar{A}_{F_n} = \mathcal{O}_{F_n}/\mathfrak{P}_{F_n}^{r_n}$, $f(y_n)_{F_n} = f_n(x_n) = 0$. Therefore, $\nu_{F_n}(f(y_n)_{F_n}) \geq r_n$ and hence

$$\nu_{X_K(L)}(f(y_n)) \geq \frac{1}{d}\nu_{X_K(L_n)}(f(y_n)) \geq \frac{r_n}{d} \geq \frac{p-1}{dp}i(L_n/F_n) = \frac{p-1}{dp}\psi_{F_n/E_n}(i(L/E_n)) \geq \frac{p-1}{dp}i(L/E_n).$$

Then the claim follows from that $\lim_{n\to+\infty} i(L/E_n) = +\infty$. □

Remark 3.3. Let L_n be as above. By Proposition 3.7 (2), we know that for $n \gg 0$, L_n's are isomorphic to each other such that $[L_n : L] = d$. Since $\sharp(\mathrm{Hom}_L(L_n, \bar{K})) = [L_n : L]$, by replacing L_n's by a certain subsequence, we may assume $L_1 = L_2 = \cdots =: M$. Then $[M : L] = d$ such that $X_K(M) = X$.

4 Ramification theory

Let L/K be an infinite APF extension. We study the ramification theory of extensions of $X_K(L)$ in this section.

Definition 4.1. Let σ be an automorphism of a local field X and $\pi \in \mathcal{O}_X$ be a uniformizer. Define

$$i_X(\sigma) = \begin{cases} \nu_X(\frac{\sigma(\pi)}{\pi} - 1), & \text{if } \sigma \text{ acts on } k_X \text{ trivially} \\ -1, & \text{else} \end{cases}.$$

Let G be a group which acts on X. Then for any $u \geq -1$, define $G_u = \{\sigma \in G \mid i_X(\sigma) \geq u\}$.

4.1 Ramification theory of $X_K(L)$

From now on, let $X = X_K(L)$ and we equip $\mathrm{Aut}(X) = \{\sigma : X \to X \mid \sigma \text{ is continuous}\}$ with the topology induced by $\{\mathrm{Aut}(X)_u\}_{u\geq -1}$.

Proposition 4.2. *Let σ be a K-automorphism of L. Then there exists an $E \in \mathcal{E}_{L/K}$ such that for any $F \in \mathcal{E}_{L/E}$, $i_F(\sigma) = i_X(X_K(\sigma))$.*

We first give some interesting applications of this proposition.

Lemma 4.3. *For any finite Galois extension L'/L and any $E' \in \mathcal{E}_{L'/K}$ such that $L' = LE'$, there exists an $F' \in \mathcal{E}_{L'/E'}$ such that*

(1) $L' = LE'$;

(2) Put $F = F' \cap L$, then F'/F is finite Galois with $\mathrm{Gal}(F'/F) \cong \mathrm{Gal}(L'/L)$*;*

(3) For any $u \geq -1$, we have $\mathrm{Gal}(F'/F)_u \cong \mathrm{Gal}(X_K(L')/X)_u$.

Proof. Let $E_0 \in \mathcal{E}_{L'/K}$ such that for any $\sigma \in \mathrm{Gal}(L'/L)$ and any $E \in \mathcal{E}_{L'/E_0}$, $i_{X_K(L')}(X_K(\sigma)) = i_E(\sigma)$. Let $E_1 = E'E_0$ and $F' = \prod_{\sigma \in \mathrm{Gal}(L'/L)} \sigma(E_1)$. We claim F' satisfies all desired conditions:

For (1): Since $E' \subset F'$, we have $LF' = L'$.

For (2): Clearly, $\mathrm{Gal}(L'/L)$ acts on F'. We claim this action is faithful: Indeed, for any $\sigma \in \mathrm{Gal}(L'/L)$, since $E_0 \subset F'$, we have $i_{F'}(\sigma) = i_{X_K(L')}(X_K(\sigma))$. So σ acts on F' trivially if and only if $X_K(\sigma) = 1$, which happens exactly when $\sigma = 1$.

Now, the second condition follows from that $F = F' \cap L = F = (F')^{\mathrm{Gal}(L'/L)}$ and Proposition 3.4 (i.e. $\mathrm{Gal}(L'/L) \cong \mathrm{Gal}(X_K(L')/X)$).

For (3): This follows from that $i_{F'}(\sigma) = i_{X_K(L')}(X_K(\sigma))$ directly. □

Corollary 4.4. *Assume L/K is Galois and define $G = \mathrm{Gal}(L/K)$. Then G acts on X faithfully whose topology is compatible with that of $\mathrm{Aut}(X)$. More precisely, we can identify the ramification groups*

$$\mathrm{Gal}(L/K)^u = G_{\psi_{L/K}(u)} = \{\sigma \in G \mid i_X(X_K(\sigma)) \geq \psi_{L/K}(u)\}.$$

Proof. The faithfulness of G-action on $X_K(L)$ can be confirmed as in the proof of Proposition 3.4: Let $\sigma \in G$ such that $X_K(L)$ act on X trivially. Then it acts on $k_{X_K(L)} = k_L$ trivially. Let $\pi = (\pi_E)_{E \in \mathcal{E}_{L/K_1,G}}$ be a uniformizer of X, where $\mathcal{E}_{L/K_1,G} = \{E \in \mathcal{E}_{L/K_1} \mid E/K \text{ is Galois}\}$. Then π_E is also a uniformizer of E. Since $X_K(\sigma)(\pi) = \pi$, we have $\sigma(\pi_E) = \pi_E$ for all E. So $\sigma = 1$.

For any $\sigma \in G$, let $E_\sigma \in \mathcal{E}_{L/K}$ be as in Proposition 4.2 and $\mathcal{E}_\sigma := \mathcal{E}_{L/E_\sigma} \cap \mathcal{E}_{L/K_1,G}$. Then we have

$$\mathrm{Gal}(L/K)^u = \varprojlim_{E \in \mathcal{E}_\sigma} \mathrm{Gal}(E/K)^u \text{ and } \mathrm{Gal}(E/K)^v = \mathrm{Gal}(E/K)_{\psi_{E/K}(v)}.$$

By Proposition 4.2, $i_X(X_K(\sigma)) = i_E(\sigma)$ for any $E \in \mathcal{E}_\sigma$. Therefore

$$X_K(\sigma) \in G_{\psi_{L/K}(u)} \Leftrightarrow \sigma \in \mathrm{Gal}(E/K)_{\psi_{L/K}(u)} = \mathrm{Gal}(E/K)^{\phi_{E/K}(\psi_{L/K}(u))},\ \forall\, E \in \mathcal{E}_\sigma \Leftrightarrow \sigma \in \mathrm{Gal}(L/K)^u,$$

where the second equivalence follows from that for a fixed $u \geq -1$, $\lim_{E \to L} \phi_{E/K}(\psi_{L/K}(u)) = u$. □

Corollary 4.5. *Let M/K be a Galois extension of K containing L. Then the isomorphism $\mathrm{Gal}(M/L) \cong \mathrm{Gal}(X_{L/K}(M)/X)$ preserves ramifications in the following sense: For any $u \geq -1$,*

$$\mathrm{Gal}(X_{L/K}(M)/X)^u = \mathrm{Gal}(M/L)^u (:= \mathrm{Gal}(M/K)^{\phi_{L/K}(u)} \cap \mathrm{Gal}(M/L)).$$

In particular, by taking $M = \bar{K}$ and applying Theorem 3.6, we have

$$G_X^u = G_L^u (:= G_L \cap G_K^{\phi_{L/K}(u)}).$$

Proof. Fix a $u \geq -1$ and a $K_u \in \mathcal{E}_{L/K}$ such that for any $E \in \mathcal{E}_{L/K_u}$, $\phi_{E/K}(u) = \phi_{L/K}(u)$. Let $E_1 \subset E_2 \subset \cdots \subset M$ be finite Galois extensions of K containing K_u such that $\cup_{n \geq 1} E_n = M$. Put $L_n = LE_n$ and then they are finite Galois over L. By Lemma 4.3, one can find $F_n \subset \mathcal{E}_{L_n/E_n}$ such that (1) $LF_n = L_n$ and (2) $F_n/L\cap F_n$ is finite Galois with Galois group $\mathrm{Gal}(F_n/L\cap F_n) \cong \mathrm{Gal}(L_n/L)$ such that for any $u \geq -1$, $\mathrm{Gal}(F_n/L\cap F_n)_u \cong \mathrm{Gal}(X_K(L_n)/X)_u$. In particular, L and F_n are linearly disjoint over $L \cap F_n$ and $\psi_{X_K(L_n)/X} = \psi_{F_n/F_n\cap L}$.

Since $\mathrm{Gal}(X_{L/K}(M)/X)^u = \varprojlim \mathrm{Gal}(X_K(L_n)/X)^u$, $\sigma \in \mathrm{Gal}(X_{L/K}(M)/X)^u$ if and only if for any $n \geq 1$, $i_{X_K(L_n)}(\sigma) \geq \psi_{X_K(L_n)/X}(u)$; equivalently, for any $n \geq 1$, $i_{F_n}(\sigma) \geq \psi_{F_n/F_n\cap L}(u)$. Since

$$\psi_{F_n/F_n\cap L}(u) = \psi_{F_n/K}(\phi_{F_n\cap L/K}(u)) = \psi_{F_n/K}(\phi_{L/K}(u)) \ (\because K_u \subset E_n \subset F_n \cap L),$$

$\sigma \in \mathrm{Gal}(X_{L/K}(M)/X)^u$ if and only if $\sigma \in \mathrm{Gal}(F_n/K)_{\psi_{F_n/K}(\phi_{L/K}(u))} = \mathrm{Gal}(F_n/K)^{\phi_{L/K}(u)}$ for any $n \geq 1$; equivalently, $\sigma \in \mathrm{Gal}(M/K)^{\phi_{L/K}(u)} \cap \mathrm{Gal}(M/L) = \mathrm{Gal}(M/L)^u$, because $\cup_{n\geq 1} F_n = M$. □

Corollary 4.6. *Let M/L be a separable algebraic extension. Then M/K is APF if and only if $X_{L/K}(M)/X$ is. If this is the case and moreover M/L is infinite, then there exists a canonical isomorphism $X_K(M) \cong X_X(X_{L/K}(M))$.*

Proof. By Corollary 4.5, we have

$$\begin{aligned}&[G_X : G_X^u G_{X_{L/K}(M)}] = [G_L : G_L^u G_M] = [G_L : (G_K^{\phi_{L/K}(u)} \cap G_L)G_M] \\ =&[G_L : G_K^{\phi_{L/K}(u)} G_M \cap G_L] = [G_K^{\phi_{L/K}(u)} G_L : G_K^{\phi_{L/K}(u)} G_M].\end{aligned}$$

Since $[G_K : G_K^{\phi_{L/K}(u)} G_L] < +\infty$ (as L/K is APF), $G_X^u G_{X_{L/K}(M)}$ is open in G_X if and only if $G_K^{\phi_{L/K}(u)} G_M$ is open in G_K. So $X_{L/K}(M)$ is APF if and only if M/K is so.

It remains to construct an isomorphism $j : X_K(M) \xrightarrow{\cong} X_X(X_{L/K}(M))$. We remark that $\mathcal{E}_{M/L} = \mathcal{E}_{X_{L/K}(M)/X}$ by Proposition 3.7 (2).

For any $\underline{x} = (x_E)_{E\in\mathcal{E}_{M/K}} \in X_K(M)$ and for any $F \in \mathcal{E}_{M/L}$, define $x_F \in X_K(F)$ by $x_F = (x_E)_{E\in\mathcal{E}_{F/K}}$. We claim that $(x_F)_{F\in\mathcal{E}_{M/L}}$ defines an element in $X_X(X_{L/K}(M))$. Indeed, for any $F \subset F'$ in $\mathcal{E}_{M/L}$, by Lemma 3.1, one can find extensions E'_n/E_n such that (1) E'_n/E_n and F/E_n are linearly disjoint; (2) $F' = FE'_n$; and (3) $F = \cup_{n\geq 1} E_n$. In particular, we have $x_{F'} = (x_{E'_n})_{n\geq 1}$ and $x_F = (x_{E_n})_{n\geq 1}$. By functoriality of X_K, we see that

$$N_{X_K(F')/X_K(F)}(x_{F'}) = (N_{F'/F}(x_{E'_n}))_{n\geq 1} = (N_{E'_n/E_n}(x_{E'_n}))_{n\geq 1} = (x_{E_n})_{n\geq 1} = x_F.$$

So we get a morphism $j : X_K(M) \to X_X(X_{L/K}(M))$ sending $\underline{x}$ to $(x_F)_F$, which obviously preserves multiplications.

The map j is clearly injective and we now show that it is also surjective. Indeed, for any $(x_F)_{F\in\mathcal{E}_{M/L}} \in X_X(X_{L/M}(K))$, we write $x_F = (x_{F,E})_{E\in\mathcal{E}_{F/K}}$. We claim that $x_{F,E} = x_{F',E}$ when $F \subset F'$. To conclude, it suffices to consider a special sequence $E_n \in \mathcal{E}_{F/K}$ such that $F = \cup_{n\geq 1} E_n$ (because if $E \subset E_n$, then $x_{F,E} = N_{E_n/E}(x_{F,E_n}) = N_{E_n/E}(x_{F',E_n}) = x_{F',E}$). So we may choose E'_n/E_n as above and then get

$$(x_{F',E_n})_{n\geq 1} = (N_{E'_n/E_n}(x_{F',E'_n}))_{n\geq 1} = N_{X_K(F')/X_K(F)}(x_{F'}) = x_F = (x_{F,E_n})_{n\geq 1}.$$

It remains to show that j also preserves additions; that is, for any $\underline{x} \in X_K(M)$, $j(\underline{x}+1) = j(\underline{x})+1$. Let $\underline{y} = \underline{x} + 1$ and then $y_E = \lim_{E' \to M} N_{E'/E}(x_{E'} + 1)$ for any $E \in \mathcal{E}_{M/K}$. Therefore,

$$y_F = (\lim_{E' \to M} N_{E'/E}(x_E + 1))_{E \in \mathcal{E}_{F/K}}.$$

On the other hand, let $\underline{z} = j(\underline{x}) + 1$, then for any $F \in \mathcal{E}_{M/L}$, we have

$$z_F = \lim_{F' \to M} N_{X_K(F')/X_K(F)}(x_{F'} + 1).$$

So we have to show that $y_F = z_F$.

We claim that $\lim_{F \to M} \nu_{X_K(F)}(y_F - 1 - x_F) = +\infty$. Granting this, for any F'/F, we have

$$\begin{aligned}
\nu_{X_K(F)}(N_{X_K(F')/X_K(F)}(1 + x_{F'}) - y_F) &= \nu_{X_K(F)}(N_{X_K(F')/X_K(F)}(1 + x_{F'}) - N_{X_K(F')/X_K(F)}(y_{F'})) \\
&\geq \phi_{X_K(F')/X_K(F))}(\nu_{X_K(F')}(x_{F'} + 1 - y_{F'})) \quad (\because \text{Proposition } 2.13) \\
&\geq \phi_{X_K(M)/X_K(F))}(\nu_{X_K(F')}(x_{F'} + 1 - y_{F'})).
\end{aligned}$$

By letting $F' \to M$, we get $z_F = y_F$ as desired.

It remains to confirm $\lim_{F \to M} \nu_{X_K(F)}(y_F - 1 - x_F) = +\infty$. In other words, for any $A > 0$, we have to find an $F \in \mathcal{E}_{M/L}$ such that for any $F' \in \mathcal{E}_{M/F}$, $\nu_{X_K(F')}(y_{F'} - 1 - x_{F'}) \geq A$. Let $E \in \mathcal{E}_{M/K}$ such that $\frac{p-1}{p} i(M/E) \geq A$ and define $F = EL$. For any $F' \in \mathcal{E}_{M/F}$, as F'/E is totally ramified, we have

$$\begin{aligned}
\nu_{X_K(F')}(y_{F'} - 1 - x_{F'}) &= \nu_E((y_{F'} - 1 - x_{F'})_E) \\
&= \nu_E(\lim_{E' \to F'} N_{E'/E}(y_{F',E'} - 1 - x_{F',E'})) \\
&= \nu_{E'}(y_{F',E'} - 1 - x_{F',E'}) \\
&= \nu_{E'}(\lim_{E'' \to F'} N_{E''/E'}(1 + x_{F',E''}) - 1 - \lim_{E'' \to F'} N_{E''/E'}(x_{F',E''})) \\
&\geq \frac{p-1}{p} i(M/E') \quad (\because \text{Proposition } 2.7) \\
&\geq A.
\end{aligned}$$

The proof is complete. □

4.2 Proof of Proposition 4.2

The rest of this section is devoted to proving that for any $\sigma \in \mathrm{Aut}_K(L)$, there exists an $E \in \mathcal{E}_{L/K}$ such that for any $F \in \mathcal{E}_{L/E}$, $i_F(\sigma) = i_X(X_K(\sigma))$. The $\sigma = 1$ case is trivial and hence we assume $\sigma \neq 1$. Moreover, if $i_X(X_K(\sigma)) = -1$, then σ acts on $k_{X_K(L)} \cong k_L$ non-trivially. In this case, we may choose $E = K_1$ (which implies that $k_F = k_L$ for any $\mathcal{E}_{L/E}$).

From now on, we assume $i_X(X_K(\sigma)) \geq 0$ and there exists an $E_0 \in \mathcal{E}_{L/K_1}$ such that $0 < i_{E_0}(\sigma) < +\infty$.

Lemma 4.7. *For any* $E \in \mathcal{E}_{L/K}$*,* $i_E(\sigma) \leq \psi_{L/E_0}(i_{E_0}(\sigma))$*.*

Proof. For any $E \in \mathcal{E}_{L/E_0}$, let $j(\sigma) = \sup_{\sigma' \mapsto \sigma} i_E(\sigma')$. By Lemma 1.4 (2), we have $i_E(\sigma) = \phi_{E/E_0}(j(\sigma))$; equivalently, $j(\sigma) = \psi_{E/E_0}(i_{E_0}(\sigma))$. So we have $i_E(\sigma) \leq j(\sigma) \leq \psi_{L/E_0}(i_{E_0}(\sigma))$ as desired. □

Now, let $E \in \mathcal{E}_{L/E_0}$ such that $i(L/E) > \psi_{L/E_0}(i_{E_0}(\sigma))$. Then L/E is totally widely ramified.

Lemma 4.8. *For any* $F \in \mathcal{E}_{L/E}$*,* $i_F(\sigma) = i_E(\sigma)$*.*

Proof. Let F'/K be the Galois closure of F/K and $G = \mathrm{Hom}_{\sigma(E)}(\sigma(F), F')$. Then $\sharp G = [F : E]$. Since F/E is totally ramified, by Lemma 1.2 (1), we have

$$i_E(\sigma) = \frac{1}{[F:E]} \sum_{\sigma' \mapsto \sigma} i_F(\sigma') = \frac{1}{[F:E]} \sum_{\tau \in G} i_F(\tau\sigma).$$

By Lemma 1.6 (2), $i_{\sigma(F)}(\tau) \geq i(\sigma(F)/\sigma(E)) = i(F/E) \geq i(L/E)$. On the other hand, let π be a uniformizer of F, then we have

$$\frac{\tau\sigma(\pi)}{\tau(\pi)} - 1 = \frac{\tau\sigma(\pi)}{\pi}(\frac{\sigma(\pi)}{\pi} - 1 + 1)^{-1} - 1 = \frac{\tau\sigma(\pi)}{\pi} - 1 + \frac{\tau\sigma(\pi)}{\pi}\sum_{n\geq 1}(\frac{\sigma(\pi)}{\pi} - 1)^n.$$

Since $\nu_F(\frac{\sigma(\pi)}{\pi} - 1) = i_F(\sigma) \leq \psi_{L/E_0}(i_{E_0}(\sigma)) < i(L/E) \leq i_{\sigma(F)}(\tau) = \nu_{\sigma(F)}(\frac{\tau\sigma(\pi)}{\tau(\pi)} - 1)$, we must have

$$i_F(\tau\sigma) = \nu_F(\frac{\tau\sigma(\pi)}{\pi} - 1) = \nu_F(\frac{\sigma(\pi)}{\pi} - 1) = i_F(\sigma),$$

which implies that $i_E(\sigma) = \frac{\sharp G}{[F:E]} i_F(\sigma) = i_F(\sigma)$.

Finally, we show that $i_F(\sigma) = i_X(X_K(\sigma))$ for any $F \in \mathcal{E}_{L/E}$. By the above lemma, it suffices to find an $F \in \mathcal{E}_{L/E}$ such that $i_F(\sigma) = i_X(X_K(\sigma))$. Let $K_0 \subset K_1 \subset \cdots$ be the elementary chain of L/K and then for any $n \gg 0$, we have (1) $E \subset K_n$ and (2) $r(K_n) \geq \frac{p-1}{p} i(K_n/K) > \psi_{L/E_0}(i_{E_0}(\sigma)) + 1 \geq i_{K_n}(\sigma) + 1$. We remark that $\sigma(K_n) = K_n$ for any n (as $\sigma(L) = L$) by the uniqueness of K_n's.

Lemma 4.9. *For* $n \gg 0$*,* $i_{K_n}(\sigma) = i_X(X_K(\sigma))$*.*

Proof. Let $\underline{pi} = (\pi_n)_{n \gg 0}$ be a uniformizer of X with π_n a uniformizer of K_n for each $n \gg 0$. Then

$$i_X(X_K(\sigma)) = \nu_X(X_K(\sigma)(\underline{\pi}) - \underline{\pi}) - 1 = \nu_{K_n}((\sigma(\underline{\pi}) - \underline{\pi})_{K_n}) - 1 = \nu_{K_n}(\lim_{m \to +\infty} N_{K_m/K_n}(\sigma(\pi_m) - \pi_m)) - 1.$$

By Proposition 2.7, we know that

$$\nu_{K_n}(\lim_{m \to +\infty} N_{K_m/K_n}(\sigma(\pi_m) - \pi_m) - (\sigma(\pi_n) - \pi_n)) \geq r(K_n).$$

As $\nu_{K_n}(\sigma(\pi_n) - \pi_n) = i_{K_n}(\sigma) + 1 < r(K_n)$, we must have

$$\nu_{K_n}(\lim_{m \to +\infty} N_{K_m/K_n}(\sigma(\pi_m) - \pi_m)) = \nu_{K_n}(\sigma(\pi_n) - \pi_n) = i_{K_n}(\sigma) + 1.$$

So we deduce that $i_X(X_K(\sigma)) = i_{K_n}(\sigma)$ as expected. □

The proof of Proposition 4.2 is complete. □

5 Infinite SAPF extensions are perfectoid

Using fancy language, the goal of this section is to show the following result:

Theorem 5.1. *Each infinite SAPF extension L/K has $\hat{L}$ as a perfectoid field in the sense of [Sch] such that the complete radical closure $\widehat{X}_K(L)_r$ of $X_K(L)$ is canonically isomorphic to $\hat{L}^\flat$, the tilting of $\hat{L}$ in the sense of [Sch].*

5.1 The tilting functor

In this part, let C be a complete valuation field with perfect residue field k_C of characteristic p.

Construction 5.2. *Let $C^\flat = \varprojlim_{x \mapsto x^p} C$ and $A_C := \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p$. For any $\underline{x} = (x_n)_{n \geq 0} \in C^\flat$, define $\nu(\underline{x}) = \nu_C(x_0)$ and let $\mathcal{O}_{C^\flat} = \{\underline{x} \mid \nu(\underline{x}) \geq 0\}$. Then $\mathcal{O}_{C^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_C$.*

For any $0 \neq a = (a_n)_{n \geq 0} \in A_C$, let $m \geq 0$ such that $a_m \neq 0$ and $\tilde{a}_m \in \mathcal{O}_C$ be a lifting of a_m. Then $p^m \nu_C(\tilde{a}_m)$ only depends on $\underline{a}$ and we denote this value by $\nu(\underline{a})$.

Clearly, there exists a morphism $\iota : \mathcal{O}_{C^\flat} \to A_C$ of monoids by sending $(x_n)_{n \geq 0}$ to $(x_n \mod p)_{n \geq 0}$. Clearly, ι preserves ν.

For any $x \in k_C$ with Teichmüller lifting $[x] \in \mathcal{O}_C$, the element $([x^{\frac{1}{p^n}}])_{n \geq 0}$ is well-defined in $\mathcal{O}_{C^\flat}$, which induces a morphism $f_C : k_C \to \mathcal{O}_{C^\flat}$ of monoids.

We first show that $C^\flat$ is a field. The idea is similar to the proof of Proposition 2.3.

Proposition 5.3. *(1) For any $\underline{x} = (x_n)_{n \geq 0}, \underline{y}_{n \geq 0} \in \mathcal{O}_C^\flat$ and any $n \geq 0$, $\{(x_{n+m} + y_{n+m})^{p^m}\}_{m \geq 0}$ converges to a unique element $z_n \in \mathcal{O}_C$. As a consequence, $\underline{z} = (z_n)_{n \geq 0}$ is a well-defined element in $\mathcal{O}_{C^\flat}$ and we denote it by $\underline{x} + \underline{y} := \underline{z}$.*

(2) For any $\underline{x} = (x_n)_{n \geq 0}, \underline{y}_{n \geq 0} \in C^\flat$ and any $n \geq 0$, $\{(x_{n+m} + y_{n+m})^{p^m}\}_{m \geq 0}$ converges to a unique element $z_n \in C$. As a consequence, $\underline{z} = (z_n)_{n \geq 0}$ is a well-defined element in $\mathcal{O}_{C^\flat}$ and we denote it by $\underline{x} + \underline{y} := \underline{z}$.

(3) Under the addition defined in (2), (C, ν) is a valuation field with ring of integers $\mathcal{O}_{C^\flat}$.

Proof. Item (2) is a consequence of (1) by assuming $\nu(\underline{x}) \geq \nu(\underline{y})$ with $\underline{x} \neq 0$ and replacing $\underline{x}, \underline{y}$ by $\frac{\underline{x}}{\underline{y}}$ and 1. By definition of ν, it makes $\mathcal{O}_{C^\flat}$ a valuation ring. Then Item (3) follows as $C^\flat \setminus \{0\}$ is a group.

For (1): Since $x_{n+m}^{p^m} = x_n, y_{n+m}^{p^m} = y_n$ for any $n, m \geq 0$, we know that

$$(x_{n+m+1} + y_{n+m+1})^p \equiv x_{n+m+1}^p + y_{n+m+1}^p = x_{n+m} + y_{n+m} \mod p.$$

The following lemma is well-known:

Lemma 5.4. *Let R be a ring, I be an ideal, $x, y \in R$ and $m \geq 1$. If $x \equiv y \mod I$, then $x^{p^m} \equiv y^{p^m} \mod (p^m I, p^{m-1} I^p, \cdots, I^{p^m})$. In particular, when $I = (p)$, $x^{p^m} \equiv y^{p^m} \mod p^{m+1}$.*

In particular, we have $(x_{n+m+1}+y_{n+m+1})^{p^{m+1}} \equiv (x_{n+m}+y_{n+m}^{p^m}) \mod p^{m+1}$. This implies (1). $\square$

Theorem 5.5. *The field $C^\flat$ is a complete valuation field with respect to ν such that $\iota : \mathcal{O}_{C^\flat} \to A_C$ is an isomorphism of valuation rings and that $f_C : k_C \to \mathcal{O}_{C^\flat}$ is a ring homomorphism identifying k_C with $k_{C^\flat}$. In particular, $C^\flat$ is perfect of characteristic p.*

Proof. Clearly, (A_C, ν) is a complete valuation ring of characteristic p with residue field $\varprojlim_{x\mapsto x^p} k_C \cong k_C$. It is enough to show ι is an isomorphism. We may proceeding as the proof of Lemma 2.16.

We first show that ι preserves additions. Let $\underline{x}, \underline{y} \in \mathcal{O}_{C^\flat}$ with $\underline{z} = \underline{x} + \underline{y}$. Then we have

$$z_n = \lim_{m\to+\infty} (x_{n+m} + y_{n+m})^{p^m}.$$

Taking reductions modulo p, we have

$$\bar{z}_n = \lim_{m\to+\infty} (\bar{x}_{n+m} + \bar{y}_{n+m})^{p^m} = \bar{x}_{n+m}^{p^m} + \bar{y}_{n+m}^{p^m} = \bar{x}_n + \bar{y}_n,$$

which is exactly what we want.

Since ι preserves ν, it is an injection. We need to show it is also a surjection. Let $\underline{a} = (a_n)_{n\geq 0} \in A_C$ and let $\tilde{a}_n$ be a lifting of a_n in $\mathcal{O}_C$ for each n. The same proof for Proposition 5.3 (1) shows that for any $n \geq 0$, $\{\tilde{a}_{n+m}^{p^m}\}_{m\geq 0}$ converges to a unique element $x_n \in \mathcal{O}_C$. It is easy to see that $(x_n)_{n\geq 0}$ defines an element $\underline{x}$ in $\mathcal{O}_{C^\flat}$ such that $\iota(\underline{x}) = \underline{a}$.

Since $\mathrm{F}((x_{n+1})_{n\geq 1}) = (x_{n+1}^p)_{n\geq 1} = (x_n)_{n\geq 1}$, we see that the absolute Frobenius map F is an automorphism of $C^\flat$. $\square$

Remark 5.1. From the proof, it is easy to see that for any non-maximal closed ideal $I \in \mathcal{O}_C$ containing p, we always have $\mathcal{O}_{C^\flat} = \varprojlim_{x\mapsto x^p} \mathcal{O}_C/I$.

Definition 5.6. We call the field $C^\flat$ the **tilting** of C. In [Win], $C^\flat$ is denoted by $R(C)$.

Construction 5.7. *For any $\underline{x} = (x_n)_{n\geq 0} \in \mathcal{O}_C^\flat$, we define $\underline{x}^\sharp = x_0 \in \mathcal{O}_C$. Then there exists a ring homomorphism*

$$\theta : \mathrm{W}(\mathcal{O}_{C^\flat}) \to \mathcal{O}_C$$

sending each $\sum_{n\geq 0} p^n[\underline{x}_n]$ to $\sum_{n\geq 0} p^n \underline{x}_n^\sharp$. By the universal property of Witt vectors, this map is induced by the natural projection

$$\mathcal{O}_{C^\flat} = \varprojlim_{x\mapsto x^p} \mathcal{O}_C/p \to \mathcal{O}_C/p.$$

We say C is **perfectoid**, *if θ is surjective with kernel* $\mathrm{Ker}(\theta)$ *principle generated by an element of the form $\xi = [\underline{x}_0] + p[\underline{x}_1] + \cdots$ such that $\nu(\underline{x}_0) > 0$ and $\nu(\underline{x}_1) = 0$. We say such an element ξ is* **distinguished**.

5.2 The tiltings of infinite SAPF extensions

From now on, we always assume L/K is an infinite SAPF extension and $X = X_K(L)$. For any $n \geq 0$, let $\mathcal{E}_n = \{E \in \mathcal{E}_{L/K_1} \mid p^n | q_E := [E : K_1]\}$. Then $\mathcal{E}_n$ is cofinal in $\mathcal{E}_{L/K}$.

Proposition 5.8. *For any $\underline{x} = (x_E) \in X$ and any $n \geq 1$, $\{x_E^{p^{-n}q_E}\}_{E\in\mathcal{E}_n}$ converges to a unique element $x_n \in \hat{L}$ such that $(x_n)_{n\geq 0} \in \hat{L}^\flat$. Moreover the map $\underline{x} \to (x_n)_{n\geq 0}$ induces a continuous homomorphism $\Lambda_{L/K} : X_K(L) \to \hat{L}^\flat$.*

Remark 5.2. It is not hard to check that $\Lambda_{L/K}$ defined above preserves the valuations.

We need the following lemma:

Lemma 5.9. *Let E/K be a totally ramified separable extension of degree p^r. Then for any $x \in E$,*

$$\nu_K(\frac{N_{E/K}(x)}{x^{p^r}} - 1) \geq c(E/K).$$

Proof. Let π be a uniformizer of K. Replacing x by $\pi^n x$ for $n \gg 0$, we may assume $x \in \mathcal{O}_E$.

Let $K = K_1 \subset K_2 \subset \cdots \subset K_r = E$ be the elementary chain of E/K. We will prove the lemma by induction on r. Let $i_n = i(K_{n+1}K_n)$ and then $c(E/K_n) = \inf_{m\geq n} \frac{i_m}{[K_{m+1}:K_n]}$.

When $r = 2$, we know E/K is itself element and are reduced to show that for any $x \in \mathcal{O}_E$, $\nu_K(\frac{N_{E/K}(x)}{x^{p^r}} - 1) \geq \frac{i(E/K)}{p^r}$. Since

$$\frac{N_{E/K}(x)}{x^{p^r}} = \prod_{\sigma\in\mathrm{Hom}_K(E,\bar{K})} (1 + \frac{\sigma(x)}{x} - 1),$$

it is enough to show $\nu_E(\frac{\sigma(x)}{x} - 1) \geq i(E/K)$, as $\nu_E = [E : K]\nu_K$. Write $x = u\pi_E^r$ with $r \geq 0$ and $u \in \mathcal{O}_E^\times$ and then

$$\frac{\sigma(x)}{x} - 1 = (\frac{\sigma(\pi_E^r)}{\pi_E^r} - 1)\frac{\sigma(u)}{u} + \frac{\sigma(u)}{u} - 1.$$

By the definition of i_E, we see that $\nu_E(\frac{\sigma(x)}{x} - 1) \geq i_E(\sigma)$. Then the result follows as $i_E(\sigma) \geq \psi_{E/K}(i(E/K)) \geq i(E/K)$, by Lemma 1.8.

For $r \geq 3$ and any $x \in \mathcal{O}_E$, by inductive hypothesis, we have

$$\nu_{K_2}(\frac{N_{E/K_2}(x)}{x^{[E:K_2]}} - 1) \geq \inf_{n\geq 2} \frac{i_n}{[K_{n+1} : K_2]} = c(E/K_2),$$

which implies that

$$\nu_K(\frac{N_{E/K_2}(x)}{x^{[E:K_2]}} - 1) \geq [K_2 : K]c(E/K_2) \geq c(E/K).$$

On the other hand, we have already shown that

$$\nu_K(\frac{N_{E/K}(x)}{N_{E/K_2}(x)^{[K_2:K]}} - 1) \geq c(K_2/K) \geq c(E/K).$$

Then the lemma follows from the above two inequalities as desired. □

Corollary 5.10. *For any $\underline{x} = (x_E)_{E \in \mathcal{E}_{L/K}} \in X_K(L)$ and any $n \geq 0$, $(x_E^{p^{-n}q_E})_{E \in \mathcal{E}_n}$ converges.*

Proof. We only consider the $n = 0$ case while the general case can be handled similarly. We may assume $K = K_1$ from now on to simplify the notations. So we have to show for any $C > 0$, there exists an $E \in \mathcal{E}_0 = \mathcal{E}_{L/K}$ such that for any $E' \subset E''$ in $\mathcal{E}_{L/E}$, $\nu_K(x_{E'}^{q_{E'}} - x_{E''}^{q_{E''}}) \geq C$.

Let $K = K_1 \subset K_2 \subset \cdots \subset L$ be the elementary chain of L/K. We choose an $N \gg 0$ satisfying the following condition:

(1) If $\mathrm{char}(K) = p$, then $[K_N : K] \geq \frac{A - \nu_K(x_K)}{c(L/K)}$.

(2) If $\mathrm{char}(K) = 0$, choose an N_0 such that $(N_0 + \frac{1}{p-1})\nu_K(p) \geq A - \nu_K(x_K)$, then $[K_N : K] \geq p^{N_0} \max(1, \frac{\nu_K(p)}{(p-1)c(L/K)})$.

Now we are going to show that $E = K_N$ satisfies the desired condition.

For any $E' \subset E''$ in $\mathcal{E}_{L/E}$, by Lemma 5.9, we have

$$\nu_K(\frac{x_{E'}}{x_{E''}^{q_{E''}/q_{E'}}} - 1) = q_{E'}^{-1}\nu_{E'}(\frac{N_{E''/E'}(x_{E''})}{x_{E''}^{E'':E'}} - 1) \geq q_{E'}^{-1}c(E''/E') \geq q_{E'}^{-1}c(L/K_N).$$

Here, the last inequality follows from $c(E''/E') \geq c(E''/K_N) \geq c(L/K_N)$.

Recall that $c(L/K) = \inf_{n \geq 1} \frac{i(K_{n+1}/K)}{[K_{n+1}:K]}$ and $c(L/K_N) = \inf_{n \geq N} \frac{i(K_{n+1}/K_N)}{[K_{n+1}:K_N]}$. Then we have

$$\nu_K(\frac{x_{E'}}{x_{E''}^{q_{E''}/q_{E'}}} - 1) \geq q_{E'}^{-1}[K_N : K]c(L/K).$$

Case 1: Assume $\mathrm{char}(K) = p$. By condition (1), we have

$$\begin{aligned}\nu_K(x_{E''}^{q_{E'}} - x_{E''}^{q_{E''}}) &= q_{E'}(\nu_K(\frac{x_{E'}}{x_{E''}^{q_{E''}/q_{E'}}} - 1) + \nu_K(x_{E''}^{q_{E''}/q_{E'}})) \\ &\geq [K_N : K]c(L/K) + q_{E''}\nu_K(x_{E''}) \\ &\geq A - \nu_K(x_K) + \nu_{E''}(x_{E''}) = A\end{aligned}$$

Case 2: Assume $\mathrm{char}(K) = 0$. Since $[K_N : K] \geq p^{N_0} \frac{\nu_K(p)}{(p-1)c(L/K)}$, we have

$$\nu_K(\frac{x_{E'}}{x_{E''}^{q_{E''}/q_{E'}}} - 1) \geq q_{E'}^{-1}p^{N_0}\frac{\nu_K(p)}{p-1}.$$

Recall the following fact:

Lemma 5.11 (*[Se2, Prop. 6, n° 1.7]*)**.** *Let K be a complete discrete valued p-adic field with $e_1 = \frac{\nu_K(p)}{p-1}$. Put $\lambda(n) = \inf(pn, n+e) = \begin{cases} pn, & n \leq e_1 \\ n+e, & n \geq e_1 \end{cases}$. Then $u : U_K \to U_K$ carrying each x to x^p sends U_K^n and U_K^{n+1} to $U_K^{\lambda(n)}$ and $U_K^{\lambda(n)+1}$ and hence induces a homomorphism $u_n : U_K^n/U_K^{n+1} \to U_K^{\lambda(n)}/U_K^{\lambda(n)+1}$. Moreover, u_n is surjective with $\mathrm{Ker}(u_n)$ vanishing if $n = e_1$ and cyclic of degree p if $n \neq e_1$.*

Since $q_{E'} \geq [K_N : K] \geq p^{N_0}$ (by condition (2)), by above fact, we have

$$\nu_K(\frac{x_{E'}^{q_{E'}/p^{N_0}}}{x_{E''}^{q_{E''}/p^{N_0}}} - 1) \geq p\nu_K(\frac{x_{E'}^{q_{E'}/p^{N_0+1}}}{x_{E''}^{q_{E''}/p^{N_0+1}}} - 1) \geq \cdots \geq \frac{q_{E'}}{p^{N_0}}\nu_K(\frac{x_{E'}}{x_{E''}^{q_{E''}/q_{E'}}} - 1) \geq \frac{\nu_K(p)}{p-1}.$$

By above fact again, we conclude that

$$\begin{aligned}\nu_K(x_{E'}^{q_{E'}} - x_{E''}^{q_{E''}}) &= \nu_K(\frac{x_{E'}^{q_{E'}}}{x_{E''}^{q_{E''}}} - 1) + q_{E''}\nu_K(x_{E''}) \\ &\geq \frac{\nu_K(p)}{p-1} + N_0\nu_K(p) + \nu_{E''}(x_{E''}) \\ &\geq A - \nu_K(x_K) + \nu_K(N_{E''/K}(x_{E''})) = A\end{aligned}.$$

The proof is complete by combining both two cases together. □

Proof of Proposition5.8: For any $\underline{x} = (x_E)_{E\in\mathcal{E}_{L/K}} \in X_K(L)$, let $x_n = \lim_{E\in\mathcal{E}_n} x_E^{p^{-n}q_E}$. Since $\mathcal{E}_{n+1} \subset \mathcal{E}_n$, we have $x_{n+1}^p = x_n$ and hence get an element $\Lambda_{L/K}(\underline{x}) \in \hat{L}^\flat$. By construction, $\Lambda_{L/K}$ preserves multiplication and is injective. (If $\Lambda_{L/K}(\underline{x}) = 0$, then $x_0 = 0$, which implies that $x_E^{q_E} = 0 = x_E$ for sufficiently large E and hence $\underline{x} = 0$.)

It remains to prove that $\Lambda_{L/K}$ is additive. In other words, we need to show for any $\underline{x} \in \mathcal{O}_{X_K(L)}$, $\Lambda_{L/K}(\underline{x}+1) = \Lambda_{L/K}(\underline{x})+1$. Put $\underline{y} = \underline{x}+1$ and then we have to show that $y_n = \lim_{m\to+\infty}(1+x_{n+m})^{p^m}$.

For any $n, m \geq 0$, since $x_{n+m} = \lim_{E\in\mathcal{E}_n} x_E^{q_E p^{-n}}$, there exists some $r \geq n+m+1$ such that

$$(1) \qquad \nu_{K_1}(x_{n+m} - x_{K_r}^{q_{K_r}p^{-n-m}}) \geq \frac{p-1}{p}c(L/K_1).$$

By enlarging r if necessary, we may also requiring that

$$(2) \qquad \nu_{K_1}(y_{n+m} - y_{K_r}^{q_{K_r}p^{-n-m}}) \geq \frac{p-1}{p}c(L/K_1).$$

By noting that $\mathcal{O}_{X_K(L)} = \varprojlim_{E\in\mathcal{E}_{L/K_1}} \mathcal{O}_E/\mathfrak{P}_E^{r(E)}$, we have

$$\nu_{K_r}(y_{K_r} - 1 - x_{K_r}) \geq r(K_r) = \frac{p-1}{p}i(L/K_r) = \frac{p-1}{p}i(K_{r+1}/K_r),$$

which implies that

$$(3) \qquad \nu_{K_1}(y_{K_r} - 1 - x_{K_r}) \geq r(K_r) \geq \frac{p-1}{p}i(L/K_r) = \frac{p-1}{p}\frac{i(K_{r+1}/K_r)}{[K_r : K_1]} \geq \frac{p-1}{p}c(L/K_1).$$

Let $e = \nu_{K_1}(p)$ and $f = \frac{p-1}{p}c(L/K_1)$. By (1) and Lemma 5.4, we see that

$$(4) \qquad \nu_{K_1}((1+x_{n+m})^{p^m} - (1+x_{K_r}^{q_{K_r}p^{-n-m}})^{p^m}) \geq \inf(me+f, (m-1)e+pf, \cdots, p^m f)$$

By (2) and Lemma 5.4, we see that

$$(5) \qquad \nu_{K_1}(y_n - y_{K_r}^{q_{K_r}p^{-n}}) \geq \inf(me+f, (m-1)e+pf, \cdots, p^m f).$$

By (3) and Lemma 5.4 (and $r \geq n+m+1$), we have

$$(6) \qquad \nu_{K_1}(y_{K_r}^{q_{K_r}p^{-n}} - (1+x_{K_r})^{q_{K_r}p^{-n}}) \geq \inf(me+f, (m-1)e+pf, \cdots, p^m f).$$

Finally, as $1 + x_{K_r}^{q_{K_r}p^{-n-m}} \equiv (1+x_{K_r})^{q_{K_r}p^{-n-m}} \mod p$, we have

$$(7) \qquad \nu_{K_1}((1+x_{K_r}^{q_{K_r}p^{-n-m}})^{p^m} - (1+x_{K_r})^{q_{K_r}p^{-n}}) \geq (m+1)e.$$

Combining (4)-(7) together, we get

$$\nu_{K_1}(y_n - (1+x_{n+m})^{p^m}) \geq \inf((m+1)e, me+f, (m-1)e+pf, \cdots, p^m f).$$

As $e, f > 0$, we can conclude by letting $m \to +\infty$. □

5.3 Proop of Theorem 5.1

We now give a proof of our main theorem in this section. Since $\hat{L}^\flat$ is complete and perfect, the natural morphism $\Lambda_{L/K} : X_K(L) \to \hat{L}^\flat$ extends canonically to an embedding $\hat{X}_r := \hat{X}_K(L)_r \to \hat{L}^\flat$. The key ingredient is the following proposition:

Proposition 5.12. *The composition $\mathcal{O}_{\hat{X}_r} \to \mathcal{O}_{\hat{L}^\flat} \to \mathcal{O}_{\hat{L}}/p\mathcal{O}_{\hat{L}} = \mathcal{O}_L/p\mathcal{O}_L$ is a surjection.*

We first exhibit how to conclude our main theorem from the above proposition.

Proof of Theorem 5.1: We first show that $\mathcal{O}_{\hat{X}_r} \to \mathcal{O}_{\hat{L}^\flat}$ is an isomorphism. It suffices to show this morphism is surjective. By Proposition 5.12, we have a surjection $\mathcal{O}_{\hat{X}_r} \to \mathcal{O}_{\hat{L}}/p\mathcal{O}_{\hat{L}}$, which gives rise to the desired surjection

$$\mathcal{O}_{\hat{X}_r} \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_{\hat{X}_r} \to \varprojlim_{x \mapsto x^p} \mathcal{O}_{\hat{L}}/p\mathcal{O}_{\hat{L}} = \mathcal{O}_{\hat{L}^\flat}.$$

Now we show $\hat{L}$ is perfectoid in the sense of Construction 5.7.

Case 1: Assume $\mathrm{char}(K) = p$. In this case, $\mathcal{O}_{\hat{L}^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\hat{L}} \to \mathcal{O}_{\hat{L}}$ is a surjection. Therefore $\mathcal{O}_{\hat{L}}$ is itself perfect, which forces that $\mathcal{O}_{\hat{L}^\flat} = \mathcal{O}_{\hat{L}}$. So the natural map $\mathrm{W}(\mathcal{O}_{\hat{L}^\flat}) \to \mathcal{O}_{\hat{L}}$ is surjection with kernel principly generated by p. So $\hat{L}$ is perfectoid.

Case 2: Assume $\mathrm{char}(K) = p$. Since $\theta : W(\mathcal{O}_{\hat{L}^\flat}) \to \mathcal{O}_{\hat{L}}$ is induced by the surjection $\mathcal{O}_{\hat{L}^\flat} \to \mathcal{O}_{\hat{L}}/p\mathcal{O}_{\hat{L}}$, we know θ is itself a surjection. It remains to show $\mathrm{Ker}(\theta)$ is generated by a distinguished element.

Recall for any $\underline{x} = (x_E)_{E \in \mathcal{E}_{L/K_1}} \in X_K(L)$, $\nu_X(\underline{x}) = \nu_{K_1}(x_{K_1})$. So we have $\nu_X(X_K(L)^\times) = \mathbb{Z} = \nu_{K_1}(K_1^\times)$. Therefore, $\nu_X(\hat{X}_r^\times) = \mathbb{Z}[\frac{1}{p}] = \nu_{K_1}(L^\times)$. In particular, there exists an $x_0 \in \hat{L}^\flat$ such that $\nu_X(x_0) = \nu_{K_1}(p)$; that is, $\theta([x_0]) = -pu$ for some $u \in \mathcal{O}_{\hat{L}}$. By the surjection of θ, there exists an element $[x_1] + p[x_2] + \cdots \in \mathrm{W}(\mathcal{O}_{\hat{L}^\flat})$ lifting u along θ. Therefore, $\xi = [x_0] + p[x_1] + \cdots$ is contained in $\mathrm{Ker}(\theta)$ which is distinguished (because $\nu_{K_1}(x_1^\sharp) = 0$ as u is a unit). We are reduced to showing that $\mathrm{Ker}(\theta) = (\xi)$. Indeed, for any $y = [y_0] + p[y_1] + \cdots \in \mathrm{Ker}(\theta)$, we must have $\nu_X(y_0) \geq \nu_{K_1}(p)$. Therefore, there exists an element $z_0 \in \mathrm{W}(\mathcal{O}_{\hat{L}^\flat})$ such that $y = z_0\xi + py_1$ for some $y_1 \in \mathrm{Ker}(\theta)$. By iteration, there are z_n's such that $y \equiv z_0\xi + pz_1\xi + \cdots + p^n z_n\xi \mod p^{n+1}$. Since $\mathrm{W}(\mathcal{O}_{\hat{L}^\flat})$ is p-complete, we see that $z = z_0 + pz_1 + \cdots$ is well-defined such that $y = z\xi$. □

At last, we show Proposition 5.12

Proof of Proposition 5.12: Let $K \subset K_0 \subset K_1 \subset \cdots$ be the elementary chain of L/K. Let $c = \inf(\nu_{K_1}(p), \frac{p-1}{p}c(L/K_1))$ and $I = \{x \in \mathcal{O}_L \mid \nu_{K_1}(x) \geq c\}$. Then we know $\mathcal{O}_{\hat{L}^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\hat{L}}/I$. We first claim $\mathcal{O}_{\hat{X}_r} \to \mathcal{O}_{\hat{L}^\flat} \to \mathcal{O}_{\hat{L}}/I = \mathcal{O}_L/I$ is a surjection. For this, let us fix an $x \in \mathcal{O}_L$.

Choose $N \gg 1$ such that $x \in \mathcal{O}_{K_N}$. Then there exists an $\underline{y} = (y_E)_{E \in \mathcal{E}_{L/K_1}} \in \mathcal{O}_{X_K(L)} = A_K(L)$ such that

$$\nu_{K_N}(x - y_{K_N}) \geq r(K_N) = \frac{p-1}{p} i(K_{N+1}/K_N).$$

In particular, we have

$$\nu_{K_1}(x - y_{K_N}) =\geq \frac{p-1}{p} \frac{i(K_{N+1}/K_N)}{[K_N : K_1]} \geq \frac{p-1}{p} c(L/K_1) \geq c,$$

which implies that $x \equiv y_{K_N} \mod I$.

We are going to show that $\underline{y}^{1/[K_N:K_1]} \in X_K(L)_r$ as an element in $\mathcal{O}_{\hat{L}^\flat}$ (via $\Lambda_{L/K}$) has reduction x modulo I. Write $\Lambda_{L/K}(\underline{y}) = (y_n)_{n \geq 0}$. If $[K_N : K_1] = p^r$, we see that $\Lambda(\underline{y}^{1/[K_N:K_1]}) = (y_{n+r})_{n \geq 0}$. So we need to show that y_r is a lifting of y_{K_N} along $\mathcal{O}_{\hat{L}^\flat} \to \mathcal{O}_L/I$.

For any $n \geq N$, by Lemma 5.9, we have

$$\nu_{K_1}\Big(\frac{y_{K_{n+1}}^{[K_{n+1}:K_n]}}{y_{K_n}} - 1\Big) = [K_n : K_1]^{-1} \nu_{K_n}\Big(\frac{y_{K_{n+1}}^{[K_{n+1}:K_n]}}{y_{K_n}} - 1\Big) \geq [K_n : K_1]^{-1} c(K_{n+1}/K_n) = \frac{i(K_{n+1}/K_n)}{[K_{n+1} : K_1]} \geq c(L/K_1).$$

By definition of $\Lambda_{L/K}$, we see that

$$y_r = \lim_{n \to +\infty} y_{K_n}^{[K_n:K_1]p^{-r}} = \lim_{n \to +\infty} y_{K_n}^{[K_n:K_N]}.$$

In particular, for $n \gg 0$,

$$\begin{aligned} y_r \equiv y_{K_n}^{[K_n:K_N]} &= \Big(\frac{y_{K_n}^{[K_n:K_{n-1}]}}{y_{K_{n-1}}}\Big)^{[K_{n-1}:K_N]} y_{K_{n-1}}^{[K_{n-1}:K_N]} \\ &= \Big(\frac{y_{K_n}^{[K_n:K_{n-1}]}}{y_{K_{n-1}}}\Big)^{[K_{n-1}:K_N]} \cdots \frac{y_{K_{N+1}}^{[K_{N+1}:K_N]}}{y_{K_N}} \cdot y_{K_N} \\ &\equiv y_{K_N} \mod I. \end{aligned}$$

This implies the surjectivity of the composition $\iota : \mathcal{O}_{\hat{X}_r} \to \mathcal{O}_{\hat{L}^\flat} \to \mathcal{O}_L/I$.

Finally, we are reduced to showing ι upgrades to a surjection $\mathcal{O}_{\hat{X}_r} \to \mathcal{O}_{\hat{L}^\flat} \to \mathcal{O}_L/p$. Since $\nu_X(X_r^\times) = \mathbb{Z}[\frac{1}{p}] = \nu_{K_1}(\hat{L})$, by shrink c if necessary, we may assume there exists an $a \in X_K(L)_r$ such that $\nu_X(a) = \nu_{K_1}(\Lambda_{L/K}(a)^\sharp) = c$.

Fix an $x_0 \in \mathcal{O}_{\hat{L}}$. By what we have proved, there exists a $y_0 \in \mathcal{O}_{X_K(L)_r}$ such that $\iota(y_0) = x_0 \mod I$. In other words, there exists an x_1 such that $x_0 = \Lambda_{L/K}(y_0)^\sharp + \Lambda_{L/K}(a)^\sharp x_1$. By iteration, we have $x_0 = \sum_{m \geq 0} \Lambda_{L/K}(a^m y_m)^\sharp$. Modulo p, we get

$$x_0 \mod p = \sum_{m \geq 0} \Lambda_{L/K}(a^m y_m)^\sharp \mod p = \Lambda_{L/K}(\sum_{m \geq 0} a^m y_m)^\sharp \mod p.$$

In other words, $\Lambda_{L/K}(\sum_{m \geq 0} a^m y_m)^\sharp$ lifts x_0 along $\mathcal{O}_{\hat{X}_r} \to \mathcal{O}_{\hat{L}^\flat} \to \mathcal{O}_{\hat{L}}/p$. We are done. □

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