S noetherian regular all schemes are sep. F.T. $/S$.

$(S = \operatorname{Spec} R \quad R = \mathrm{DVR})$

I. Localized Chern classes

- $X \hookrightarrow Y \qquad Y - X \neq \phi$

locally free

- $\mathcal{E}_\cdot = \left(\mathcal{E}_n \xrightarrow{d_n} \mathcal{E}_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} \mathcal{E}_0 \right) \in C^b\left(\mathrm{Mod}^{lf}(\mathcal{O}_Y) \right)$

$\mathcal{E}_{n+1} = \mathcal{E}_{-1} = 0$

s.t. i) $d \circ d = 0$

(P): ii) $\forall i > 0, \ \mathcal{H}_i(\mathcal{E}_\cdot)_{|Y-X} = 0$

iii) $\mathcal{H}_0(\mathcal{E}_\cdot)_{|Y-X}$ is lf $rk = e \geq 0$.

Goal For $p \geq e+1$, define localized Chern classes

$$c^Y_{p,X}(\mathcal{E}_\cdot) : CH_*(Y) \longrightarrow CH_{*-p}(X)$$

$$= \underbrace{Z_*(Y)}_{\text{cycles over } Y} / \sim_{rat}$$

Let - $e_i = rk(\mathcal{E}_i)$

- $G_i = Gr_{e_i}(\mathcal{E}_i \oplus \mathcal{E}_{i-1})$

- ξ_i = tautological bundle of rk e_i on G_i

- $G = \underbrace{G_n \times_Y G_{n-1} \times_Y \cdots \times_Y G_0}$

$pr_i : G \to G_i$ projection

$$\underline{\xi} = \underline{\sum_{i=0}^{n} (-1)^i [pr_i^* \xi_i]} \in K_0(G) = K_0(\mathrm{Vect}(G))$$

$\forall\, y \in Y, \quad \forall\, \lambda \in k(y), \quad \lambda d_i(y) \in \mathcal{E}_{i-1}(y)$

$$\Gamma(\lambda d_i(y)) \in \mathcal{E}_i(y) \oplus \mathcal{E}_{i-1}(y)$$

$\Rightarrow \quad \varphi : Y \times \mathbb{A}^1 \longrightarrow G \times \mathbb{A}^1$ closed immersion.

$$(y, \lambda) \longmapsto \left(\prod_i \Gamma(\lambda d_i(y)), \lambda\right)$$

Let - $k_n = 0, \quad k_i + k_{i-1} = e_i \qquad 0 \le i \le n$

$$\left(k_i = e_{i+1} - e_{i+2} + \cdots \pm e_n, \qquad k_0 = e_0 - \underset{\sum (-1)^i e_i}{\overset{\|}{e}}\right)$$

$y \in Y - X \quad \Rightarrow \quad k_i = \dim \ker d_i(y) \ge 0$

- $H_i = Gr_{k_i}(\mathcal{E}_i)$

$H = H_n \times_Y H_{n-1} \times_Y \cdots \times_Y H_0$.

$\Rightarrow \quad \tau : H \to G$ closed imm. $\qquad H_0 \xrightarrow{q} G_0 = Y$

$$(L_n, \cdots, L_0) \longmapsto (L_n \oplus L_{n-1}, \cdots, L_1 \oplus L_0, L_0)$$

- $pr_0 : H \longrightarrow H_0$ $\qquad H_0 = Gr_{e_0-e}(\Sigma_0)$

Q_0 = universal quotient bundle over H_0
rk $= e$.

$$\boxed{\tau^*\xi = pr_0^* Q_0 \in K_0(H)}$$

- $H^\circ = H \times_Y (Y-X)$

$\psi : Y - X \longrightarrow H^\circ$

$y \longmapsto (\mathrm{Ker}\, d_n(y), \mathrm{Ker}\, d_{n-1}(y), ---, \mathrm{Ker}\, d_1(y), \mathrm{Im}\, d_1(y))$

$\Rightarrow$ <u>NON COMMUTATIVE</u> diagram

$$\begin{tikzcd}
Y \times \mathbb{A}^1 \arrow[rrr, "\varphi", "\text{cl. imm.}"'] & & & G \times \mathbb{A}^1 \arrow[r, hook] & G \times \mathbb{P}^1 & \alpha' \arrow[l, dashed] \\
(Y-X) \times \mathbb{A}^1 \arrow[u] \arrow[r] & (Y-X) \times \mathbb{P}^1 \arrow[r, "\psi \times id"] & H^\circ \times \mathbb{P}^1 \arrow[rr] & & H \times \mathbb{P}^1 \arrow[u, "\tau \times id"'] & \alpha'' \arrow[l]
\end{tikzcd}$$

Let - $\underline{\alpha \in Z_*(Y)}$

- $\alpha^\circ = \alpha_{|Y-X}$

Choose - $\alpha' \in Z_*(G \times \mathbb{P}^1)$ s.t.

$$\alpha'_{|G \times \mathbb{A}^1} = \varphi_*(\alpha \times [\mathbb{A}^1])$$

- $\alpha'' \in Z_*(H \times \mathbb{P}^1)$ s.t.

$$\alpha''_{|H^\circ \times \mathbb{P}^1} = \psi_*(\alpha^\circ) \times [\mathbb{P}^1]$$

Def $\boxed{\gamma := i_\infty^*(\alpha' - \alpha'') \in Z_{*-1}(G)}$

$$i_\infty^* : Z_*(G \times \mathbb{P}^1) \longrightarrow Z_{*-1}(G)$$

intersection with a principal divisor

Lemma 1) $\gamma \in Z_{*-1}(\underset{\substack{\parallel \\ G_X}}{G \times_Y X})$

2) γ is indep of α'

3) Another choice of α'' changes γ to $\gamma + \beta$

$\beta \in Z_*(\underset{\substack{\parallel \\ G_X \cap H}}{H_X})$.

Pf 2) $\alpha_1' = \alpha' + \rho$ $\quad \rho \in Z_*(G \times \{\infty\})$ $\quad i_\infty^* \rho = 0$

3) $\alpha_1'' = \alpha' + \rho$ $\quad \rho \in Z_*(H_X \times \mathbb{P}^1)$ $\quad i_\infty^* \rho \in Z_*(H_X \times \{\infty\})$

1) WMA $X = \emptyset$ choose α' & α'' and show $\gamma = 0$. $\square$.

Def For - $\pi : G_X \to X$

- $p \geq e+1$ $\qquad$ localized Chern classes

Refine $\boxed{c_p^Y(\xi) \cap \alpha := \pi_*(c_p(\xi) \cap X) \in CH_*(X)}$

Define $\boxed{c^{Y}_{p,X}(\mathcal{E}_\cdot) \cap \alpha := \pi_*\left(c_p(\xi) \cap \gamma\right) \in CH_*(X)}$

since $\xi|_{H_X}$ is LF $rk = e$, this is indep. of γ.

Similarly, $\forall\, h : Y' \to Y$

$$X' = Y' \times_Y X$$

$$c^{Y}_{p,X}(\mathcal{E}_\cdot) \cap (-) : \underbrace{Z_*}_{CH_*}(Y') \longrightarrow CH_{*-p}(X')$$

<u>Prop</u> This defines a <u>bivariant class</u> in $\underbrace{CH^p(X \to Y)}$,

i.e.

(1) If h is proper, $h' : X' \to X$, then

$$c^{Y}_{p,X}(\mathcal{E}_\cdot) \cap (h_*\alpha) = h'_*\left(c^{Y}_{p,X}(\mathcal{E}_\cdot) \cap \alpha\right) \in CH_{*-p}(X)$$

(2) If f is flat $\dim = d$, then

$$c^{Y}_{p,X}(\mathcal{E}_\cdot) \cap (h^*\alpha) = h'^*\left(c^{Y}_{p,X}(\mathcal{E}_\cdot) \cap \alpha\right) \in CH_{*-p+d}(X')$$

(3)
$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "i''"'] & Y' \arrow[r] \arrow[d, "i'"'] & Z' \arrow[d, "i"] \\
X \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$
reg. emb.

$$i^!\left(c^{Y}_{p,X}(\mathcal{E}_\cdot) \cap \alpha\right) = c^{Y}_{p,X}(\mathcal{E}_\cdot) \cap (i^!\alpha)$$

In part, - $c^Y_{p,X}(\mathcal{E}_\cdot)$ pass through $\sim_{rat}$

- commutes with Chern classes: $\forall\, V \in \mathrm{Vect}(Y)$

$$c^Y_{p,X}(\mathcal{E}_\cdot) \cap \big(c_m(V) \cap \alpha\big) = c_m(V_{|X}) \cap \big(c^Y_{p,X}(\mathcal{E}_\cdot) \cap \alpha\big)$$

Th 1) $X \xrightarrow{i} Y \xrightarrow{j} Z$ cl. imm.

$(\mathcal{E}_\cdot)$ satisfies (P) w.r.t. $X \to Z$.

$$\Rightarrow i_*\big(c^Z_{p,X}(\mathcal{E}_\cdot) \cap \alpha\big) = c^Z_{p,Y}(\mathcal{E}_\cdot) \cap \alpha$$

2) (Whitney) $0 \to \mathcal{E}_\cdot \to \mathcal{F}_\cdot \to \mathcal{G}_\cdot \to 0$ s.e.s.

$\mathrm{rk}\ \mathcal{H}_0(\mathcal{E}_\cdot)_{|Y-X} = e$

$\mathcal{F}_\cdot \qquad = f \qquad\qquad f = e+g$

$\mathcal{G}_\cdot \qquad = g$

Then $\forall\, p \geq f+1$

$$\boxed{c^Y_{p,X}(\mathcal{F}_\cdot) = \sum_{j=0}^{p} c'_j(\mathcal{E}_\cdot)\, c'_{p-j}(\mathcal{G}_\cdot)}$$

$$c'_j(\mathcal{E}_\cdot) = \begin{cases} c_j(\mathcal{E}_\cdot) & j \leq e \\ c^Y_{j,X}(\mathcal{E}_\cdot) & j \geq e+1 \end{cases}$$

idem for $c_j'(\mathcal{G}_\cdot)$

In other words,

$$\underbrace{c_X^Y(\mathcal{E}_\cdot)}_{} := 1 + c_1(\mathcal{E}_\cdot) + \cdots + c_e(\mathcal{E}_\cdot) + c_{e+1,X}^Y(\mathcal{E}_\cdot) + \cdots$$

Then $c_X^Y(\mathcal{F}_\cdot) = c_X^Y(\mathcal{E}_\cdot) \cdot c_X^Y(\mathcal{G}_\cdot)$

<u>Cor</u> $c_{p,X}^Y(\mathcal{E}_\cdot)$ only depends on $[\mathcal{E}_\cdot] \in D^b(\mathcal{O}_Y)$

<u>Prototype</u> $R = \mathrm{DVR}$ char $\eta = 0$

$S = \operatorname{Spec} R \leftarrow \eta$, $\leftarrow s$

X/S flat proper, $\dim = d$

$X_\eta \neq \phi$ $\boxed{X_\eta \text{ smooth}}$

X reg. integral

(X/S <u>arithmetic scheme</u>)

X_s closed fiber

$\mathcal{E}_\cdot$ = resolution of $\Omega^1_{X/S}$

$$\underbrace{c^X_{d+1, X_s}(\Omega^1_{X/S}) \cap [X]}_{} \in CH_0(X_s)$$

$$\chi^{loc}(X) := (-1)^{d+1} \deg(\ -\ )$$

localized Euler characteristic.

II. Localized intersection product

X/S sep. F.T. $X_\eta \neq \phi$

- $\mathcal{J}^\bullet = \bigoplus_{n \geq 0} \mathcal{J}^n \in Gr^{\mathbb{Z}} Alg(\mathcal{O}_X)$

s.t. - $\mathcal{J}^0 = \mathcal{O}_X$

- $\mathcal{J}^1 \in Coh(\mathcal{O}_X)$

- $\mathcal{J}^1$ generates $\mathcal{J}^\bullet$ over $\mathcal{O}_X$

- $hdim(\mathcal{J}^1) < \infty$

(i.e. $\mathcal{J}^1$ has a finite l.f. resolution)

- $\mathcal{J}^1_{|X_\eta}$ lf $rk = d$

- $Y = Spec_{\mathcal{O}_X}(\mathcal{J}^\bullet)$ cone

- $\mathbb{P} = Proj(\mathcal{J}^\bullet[z])$ projective completion

$q : \mathbb{P} \to X$

- $K := ker\left(q^*(\mathcal{J}^1 \oplus \mathcal{O}_X) \xrightarrow{\delta} \mathcal{O}_{\mathbb{P}}(1)\right)$

Let

$$0 \to \underbrace{\Sigma_n \to \Sigma_{n-1} \to \cdots \to \Sigma_0}_{\Sigma_\bullet} \xrightarrow{\varepsilon} \mathcal{J}^1 \to 0$$

be a resolution by lf $\mathcal{O}_X$-mod.

- $L := \ker\left(q^*(\mathcal{E}_0 \oplus \mathcal{O}_X) \xrightarrow{\delta_0(\mathcal{E}\oplus id)} \mathcal{O}_{\mathbb{P}}(1)\right)$

$F_\cdot := [q^*\mathcal{E}_n \to q^*\mathcal{E}_{n-1} \to \cdots \to q^*\mathcal{E}_1 \to L] \in C^b\left(\mathrm{Mod}^{lf}(\mathcal{O}_{\mathbb{P}})\right)$

Lemma $F_\cdot$ satisfies (P) w.r.t. $\mathbb{P}_S \to \mathbb{P}$

D/ $\mathcal{J}'_{|X_y}$ lf $\Rightarrow q_y$ smooth

$\Rightarrow 0 \to q^*\mathcal{E}_n \to \cdots \to q^*\mathcal{E}_1 \to q^*(\mathcal{E}_0 \oplus \mathcal{O}_X) \to q^*(\mathcal{J}' \oplus \mathcal{O}_X) \to 0$

is exact over $\mathbb{P}_y$

$\Rightarrow$ ii)

$\mathcal{H}_0(F_\cdot)_{|\mathbb{P}_y} = K_{|\mathbb{P}_y}$ lf rk $= d$ $\square$.

Def $\psi : CH_k(\mathbb{P}) \longrightarrow CH_{k-d-1}(X_S)$

$\alpha \longmapsto q_{S*}\left((-1)^{d+1} c^{\mathbb{P}}_{d+1, \mathbb{P}_S}(F_\cdot) \cap \alpha\right)$

Lemma ψ is indep of $\mathcal{E}$.

D/ $\mathcal{E}_\cdot$, $\mathcal{E}'_\cdot$ res. WMA $\mathcal{E}_\cdot$ dominates $\mathcal{E}'_\cdot$

$\mathcal{G}_\cdot := \ker(\mathcal{E}_\cdot \to \mathcal{E}'_\cdot) \in C^b(\mathrm{Mod}^{lf}(\mathcal{O}_X))$ exact

$\Rightarrow \quad 0 \to q^*(\mathcal{G}_\cdot) \to \mathcal{F}_\cdot \to \mathcal{F}_\cdot' \to 0$ exact

$\in C^b(\mathrm{Mod}^{lf}(\mathcal{O}_{\mathbb{P}}))$

$\Rightarrow [\mathcal{F}_\cdot] = [\mathcal{F}_\cdot'] = D^b(\mathcal{O}_{\mathbb{P}})$

$\Rightarrow$ same localized Chern classes. $\square$.

Similarly, $\forall h: X' \to X$, $q': \mathrm{Proj}(h^*\mathcal{J}^\cdot[z]) \to X$

define

$\psi_{X'}: CH_k(\mathrm{Proj}(h^*\mathcal{J}^\cdot[z])) \longrightarrow CH_{k-d-1}(X'_s)$

$\alpha \longmapsto q'_{s*}\left((-1)^{d+1} c^{\mathbb{P}}_{d+1,\mathbb{P}_s}(\mathcal{F}_\cdot) \cap \alpha\right)$

<u>Def</u> $X \hookrightarrow Y$ cl. imm. def. by $\mathcal{I}$ ideal sheaf

is a <u>$*$-closed imersion of codim d</u>

if $N_X Y = \mathcal{I}/\mathcal{I}^2$ satisfies - $\mathrm{hdim}(N_X Y) < \infty$

- $N_X Y|_{X_\eta}$ lf $rk = d$

<u>Ex</u> X/S arithmetic schem

$X \longrightarrow X \times_S X$ is $*$-closed imm.

- $\mathcal{S}_X Y = \bigoplus\limits_{n \geq 0} \mathcal{I}^n / \mathcal{I}^{n+1}$

$C_X Y = \mathrm{Spec}_{\mathcal{O}_X}(\mathcal{S}_X Y)$ normal cone

$\mathbb{P} = \mathrm{Proj}(\mathcal{S}_X Y[z])$ proj. completion

$\mathcal{E}_\bullet \longrightarrow N_X Y$ resolution

$\rightsquigarrow \mathcal{F}_\bullet \in C^b(\mathrm{Mod}^{lf}(\mathcal{O}_{\mathbb{P}}))$

Let $f: V \to Y$ $\quad$ $\dim V = k$

$$\begin{tikzcd} C' \arrow[d] \arrow[dr] & & \\ g^*C \arrow[r, hook] \arrow[d] & W \arrow[r, hook] \arrow[d, "g"'] & V \arrow[d, "f"] \\ C \arrow[r, hook] & X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

$\mathcal{J}$ = ideal sheaf of W in V

$$\bigoplus_{n \geq 0} g^* \mathcal{I}^n / \mathcal{I}^{n+1} \twoheadrightarrow \mathcal{S}_W V = \bigoplus_{n \geq 0} \mathcal{J}^n / \mathcal{J}^{n+1}$$

$\Longrightarrow$

$$\begin{tikzcd} \underbrace{\mathrm{Proj}(\mathcal{S}_W V[z])}_{\dim = k} \arrow[r, "j", "\text{cl. imm.}"'] & \mathrm{Proj}(g^* \mathcal{S}_X Y[z]) \arrow[r, "l"] \arrow[d, "p"'] & \mathrm{Proj}(\mathcal{S}_X Y[z]) \arrow[d, "q"] \\ & W \arrow[r, "g"'] & X \end{tikzcd}$$

$\Longrightarrow [\mathrm{Proj}(\mathcal{S}_W V[z])] \in Z_k(\mathrm{Proj}(g^* \mathcal{S}_X Y[z]))$

<u>Def</u> 1) $(X \cdot V)_{loc} := p_{s*}\left((-1)^{d+1} c^{\mathbb{P}}_{d+1, \mathbb{P}_s}(\mathcal{F}_\bullet) \cap [\mathrm{Proj}(\mathcal{S}_W V[z])] \right)$

$$(X \cdot V)_{loc} := p_{S*}\left((-1)^{\cdots} c''_{d+1, \mathbb{P}_S}(\mathcal{F}_\bullet) \cap [\mathrm{Proj}(\mathfrak{f}_W V[z])] \right)$$

$$\in CH_{k-d-1}(W_S)$$

2) $\forall\, Y' \to Y$, $X' = X \times_Y Y'$, define

$$\underline{i^!_{loc}} : Z_k(Y') \longrightarrow CH_{k-d-1}(X'_S)$$

$$\sum n_i V_i \longmapsto \sum n_i (X \cdot V_i)_{loc}$$

<u>localized Gysin homomorphism</u>

<u>Rk</u> 1) If $W_Y = \emptyset$, then

$$(X \cdot V)_{loc} = \Big\{ \underline{g^*(c(N_X Y))^{*} \cap s(W, V)} \Big\}_{k-d-1}$$

(dual)

$$\in CH_{k-d-1}(W_S) = CH_{k-d-1}(W)$$

(i.e. no need to localize, cf. Fulton Prop. 6.1 (a))

2) $i^!_{loc}$ does <u>not</u> pass through $\sim_{rat}$.

<u>Prop</u>

$$\begin{tikzcd} X'' \arrow[r, "i''"] \arrow[d, "l"'] & Y'' \arrow[d, "h"] \\ X' \arrow[r, "i'"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ {} & {} \end{tikzcd} \quad (\dagger)$$

i $*$-cl. imm. codim $= d$.

$$\check{X} \xrightarrow{i} Y$$

1) if h proper, $\alpha \in Z_k(Y'')$, then

$$i^!_{loc} h_*(\alpha) = l_{s*}(i^!_{loc}(\alpha)) \in CH_{k-d-1}(X'_s)$$

2) If h flat dim $= n$, $\alpha \in Z_k(Y')$, then

$$i^!_{loc} h^*(\alpha) = l_s^*(i^!_{loc}(\alpha)) \in CH_{k+n-d-1}(X''_s)$$

<u>Pf</u> use degree formula / flat pb. of fdl. class $\square$.

<u>Th</u> (localized Excess intersection formula) Assume $(+)$

Assume
- $i = *$-cl. imm codim $= d$
- $i' =$ reg. imm. codim $= d'$
- $J =$ ideal sheaf of i'
- $M = (J/J^2)^\vee$
- $\dim Y'' = k$

$e = d - d'$

Let
- $\Sigma_\cdot \to N_X Y$ resolution
- $F = \ker(g^* \Sigma_0 \twoheadrightarrow J/J^2)$

- $F_\cdot = \left[g^* \Sigma_n \to \cdots \to g^* \Sigma_1 \to F \right] \in C^b\left(\mathrm{Mod}^{lf}(\mathcal{O}_{X'})\right)$

<u>excess complex</u>

Then - $F_\cdot$ satisfies (P) w.r.t. $X'_S \to X'$

- $\displaystyle (X \cdot Y'')_{loc} = \sum_{j=e+1}^{d+1} (-1)^j \, c^{X'}_{j, X'_S}(F_\cdot) \cap \left\{ c(\ell^* M) \cap s(X'', Y'') \right\}_{k+j-d-1}$

$\in CH_{k-d-1}(X''_S)$

In part., if $Y' = Y''$, then

$$(X \cdot Y')_{loc} = (-1)^{e+1} \, c^{X'}_{e+1, X'_S}(F_\cdot) \cap [X'] \in CH_{k-d-1}(X'_S)$$

D/

$$\begin{tikzcd}
\mathbb{P}' = \mathrm{Proj}(\mathcal{S}_{X'} Y'[z]) \arrow[r, "j"] & \mathrm{Proj}(g^* \mathcal{S}_X Y[z]) \arrow[r, "g'"] \arrow[d, "g_1"] & X' \arrow[d] \\
 & \mathbb{P} = \mathrm{Proj}(\mathcal{S}_X Y[z]) \arrow[r, "q"] & X
\end{tikzcd}$$

Define $L = \ker\left(q^*(\Sigma_0 \oplus \mathcal{O}_X) \twoheadrightarrow \mathcal{O}_{\mathbb{P}}(1)\right)$

$K = \ker\left(q^*(N_X Y \oplus \mathcal{O}_X) \twoheadrightarrow \mathcal{O}_{\mathbb{P}}(1)\right)$

$G_\cdot = [\, q^* \Sigma \ \cdots \ q^* \Sigma \ \cdots \,] \in C^b(\mathrm{Mod}^{lf}(\mathcal{O}$

$$\mathcal{G}_\bullet = [q^*\Sigma_n \to \cdots \to q^*\Sigma_1 \to L] \in C^b(\mathrm{Mod}^{lf}(\mathcal{O}_{\mathbb{P}}))$$

$$K' = \ker\left(j^* q'^* (J/J^2 \oplus \mathcal{O}_{X'}) \twoheadrightarrow \mathcal{O}_{\mathbb{P}'}(1)\right)$$

<u>Lemma</u> $0 \to j^* q'^* F_\bullet \to j^* g_1^* \mathcal{G}_\bullet \to K' \to 0$
$\in C^b(\mathrm{Mod}^{lf}(\mathcal{O}_{\mathbb{P}'}))$

is exact.

Then compute $c^{\mathbb{P}}_{d+1,\mathbb{P}_s}(\mathcal{G}_\bullet) \cap [\mathrm{Proj}(\mathcal{J}_{X''}Y'[z])]$

using Whitney formula, (C1) & projection formula. $\square$.

<u>Cor</u>

$$\begin{tikzcd} X\times_Y V \arrow[r] \arrow[d] & V \arrow[d, "h"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

$i =$ ~~*~~-cl. imm codim $= d$

If $X\times_Y V \xrightarrow{\sim} V$

Then $(X\cdot V)_{loc} = (-1)^{d+1} c^X_{d+1,X_s}(N_X Y) \cap [V]$
$\in CH_{k-d-1}(V_s)$

In part, <u>localized self-intersection formula</u>

$$(X\cdot X)_{loc} = (-1)^{d+1} c^X_{d+1,X_s}(N_X Y) \cap [X] \in CH_{k-d-1}(X_s)$$

(generalizes Bloch's definition)

Prop (a characterization) $\underbrace{i^!_{loc}}_{}$ is the unique localized intersection product which

- is compat. with proper pf / flat pb.

- satisfies loc. EIF in codim 0 & 1

(i.e. $\forall$
$$\begin{tikzcd} W \arrow[r, "j"] \arrow[d] & V \arrow[d] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$
$\dim V = k$ (Cartier divisor)

j = iso or reg imm, codim = 1

Then $(X \cdot V)_{loc} = (-1)^{d+\varepsilon} c^{W}_{d+\varepsilon, W_s}(F_\cdot) \cap [W]$

where $-\ \varepsilon = \begin{cases} 1 & \text{if } j = \text{iso} \\ 0 & \text{else} \end{cases}$

$-\ F_\cdot$ = excess complex.)

Pf/ Blow-up. $\square$.