GERSTEN–WITT COUNTEREXAMPLES IN DIMENSION FIVE

GPT-6 ASTRA DIRECTED BY FANGZHOU JIN

Abstract. Based on recent counter-examples to the Gersten conjecture with integral, rational and finite coeffcients by Feld, Guo and OpenAI, we propose counter-examples to the Gersten conjecture for Witt groups.

1. Introduction

Based on recent counter-examples to the Gersten conjecture with integral, rational and finite coeffcients ([2], [3], [5], [8], [9]), we propose counter-examples to the Gersten conjecture for Witt groups.

More precisely, we formulate a Gersten–Witt counterexample in dimension five using a self-dual two-term complex associated with a matrix factorization of the element 3 in a ramified regular local ring. Its residue at a codimension-three prime is the nontrivial rank-one maximal Cohen–Macaulay class over the A_1-quadric cone $uv = x^2$. The class is a rank-one symmetric form, hence nonzero in the integral Witt group; it also survives with mod-2 coefficients. The determinant of the underlying MCM module is 2-torsion in the class group, but the Witt class itself is not 2-torsion.

1.1. The Gersten conjecture for Witt groups.

Let $(R, \mathfrak{m}, k)$ be a noetherian regular local ring of dimension d, with fraction field K. Throughout the classical integral statements, assume

$$2 \in R^{\times}.$$

In mixed characteristic $(0, p)$ this means $p \neq 2$. Let $\mathrm{W}(R)$ be the Witt group of nonsingular symmetric bilinear forms modulo metabolic forms. Since 2 is invertible, this is also the usual quadratic-form Witt theory.

The conjecture asserts exactness of the augmented complex

$$0 \longrightarrow \mathrm{W}(R) \longrightarrow \mathrm{W}(K) \longrightarrow \bigoplus_{x \in (\operatorname{Spec} R)^{(1)}} \mathrm{W}(\kappa(x)) \longrightarrow \cdots \longrightarrow \mathrm{W}(k) \longrightarrow 0. \tag{1}$$

We suppress one-dimensional coefficient twists: canonically, the codimension-a term uses the duality line $\operatorname{Ext}^a_{R_x}(\kappa(x), R_x)$, or the corresponding line in an equivalent duality convention. Trivializing these lines gives the displayed notation. The differentials are Witt residues with the appropriate transfer maps.

Exactness includes injectivity at the generic point, extension of unramified classes, and vanishing of all positive-degree cohomology of the Gersten–Witt complex. These are stronger claims than generic injectivity alone.

1.2. Summary of established cases.

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Hypothesis	Conclusion and reference
$\dim R \leq 4$, $2 \in R^\times$	Full exactness, Balmer–Walter [1, Corollary 10.4].
Mixed characteristic $(0,p)$, $p \neq 2$, and $p \notin \mathfrak{m}^2$	Full exactness in every dimension, Jacobson [6].
R essentially smooth over a DVR, $2 \in R^\times$	Full exactness, including hermitian versions, Gille–Panin [4].
R geometrically regular over such a DVR	Full exactness by the Popescu extension of [4].
Arbitrary regular local R, after tensoring (1) with $\mathbb{Z}[1/2]$	Exactness without an unramifiedness hypothesis; see [7, Remark 3.7].

1.3. **Discrete valuation rings and low dimension.** For a DVR V with fraction field F, residue field k of characteristic different from two, and uniformizer π, the sequence is

$$0 \longrightarrow \mathrm{W}(V) \longrightarrow \mathrm{W}(F) \xrightarrow{\partial_\pi} \mathrm{W}(k) \longrightarrow 0.$$

On one-dimensional forms, for $u \in V^\times$ and $r \in \mathbb{Z}$,

$$\partial_\pi \langle u\pi^r \rangle = \begin{cases} 0, & r \text{ even}, \\ \langle \bar{u} \rangle, & r \text{ odd}. \end{cases}$$

The choice of π trivializes the residue line. Lifting units shows surjectivity directly. Exactness identifies its kernel with forms extending over V. Completeness and henselianity are not required.

More generally, Balmer–Walter prove (1) for every regular local ring of dimension at most four with 2 invertible. Their argument uses the Gersten–Witt spectral sequence and its four-periodic pattern [1]. Thus ramified mixed-characteristic regular local rings in dimensions two, three, and four are included.

1.4. **Unramified rings and regular morphisms over DVRs.** In mixed characteristic $(0,p)$ a regular local ring is called unramified here when

$$p \notin \mathfrak{m}^2, \quad \text{equivalently, } R/pR \text{ is regular.}$$

Jacobson proves the integral Gersten–Witt conjecture in this case for $p \neq 2$, without a dimension restriction [6]. The result is Theorem 2.8 in the linked author manuscript and is cited as Theorem 3.8 in the published version. The proof relates the fundamental-ideal filtration and signature methods to Gersten results for etale cohomology.

Gille–Panin prove exactness for local rings essentially smooth over a DVR, and extend it to local rings geometrically regular over a DVR using Popescu's theorem [4]. A regular morphism here is flat with geometrically regular fibers; regularity of the source ring alone is not sufficient.

The base DVR may itself be ramified over the prime integers. For example, for an odd prime p and $e \geq 2$, let

$$V = \mathbb{Z}_p[\pi]/(\pi^e - p), \qquad R = V[t_1, \ldots, t_r]_{(\pi, t_1, \ldots, t_r)}.$$

Then R is essentially smooth over V, so the theorem applies, although $p = \pi^e \in \mathfrak{m}_R^2$. This illustrates why "unramified over the integers" and "geometrically regular over some DVR" are different hypotheses.

1.5. **Hermitian extensions.** The Gille–Panin theorem also covers Azumaya algebras with involution of the first or second kind, in the same geometric setting and with 2 invertible [4]. Its specialization to the algebra R with identity involution gives the classical theorem above. One should not transfer every classical low-dimensional theorem to arbitrary Azumaya algebras without checking additional hypotheses. Likewise, Witt groups and higher Grothendieck–Witt groups are different invariants; the exactness statements here are for the specified Gersten–Witt complexes.

1.6. **Inverting two and the remaining scope.** After tensoring the classical complex with $\mathbb{Z}[1/2]$, Gersten exactness holds for arbitrary regular local rings in the stated 2-invertible setting. Jacobson established an excellent case; the excellence restriction was subsequently removed. The precise references are recorded in [7, Remark 3.7].

It follows that any integral defect remaining in this setting is 2-primary torsion: each element of the defect is annihilated by some power of 2. This does not assert a uniform exponent. Inverting 2 in the coefficient groups is distinct from the standing assumption that 2 is already a unit in the base ring.

The theorems listed here do not settle the unrestricted integral case of ramified regular local rings of dimension at least five without an appropriate geometric regularity hypothesis. This note does not claim a counterexample in that range. In particular, a K-theory counterexample over a three-dimensional regular local ring with 2 invertible cannot be a classical Witt–Gersten counterexample over the same ring: the low-dimensional theorem already excludes it.

In mixed characteristic $(0, 2)$, the classical hypotheses above fail. Symmetric bilinear, quadratic, and derived hermitian theories must be distinguished, and their residue and devissage results must be specified separately. No general dyadic theorem or counterexample is asserted here.

2. Mixed characteristic example with 2 invertible

The self-dual complex C_A below gives a verified counterexample to Gersten injectivity in the derived Witt group, and hence, under the standard identification for regular rings with 2 invertible, to the ordinary symmetric-bilinear Gersten injectivity statement.

Let

$$R = \Big(\mathbb{Z}_{(3)}[x, u, v, t_1, t_2]/(uv - x^2 - 3)\Big)_{\mathfrak{m}}, \qquad \mathfrak{m} = (3, x, u, v, t_1, t_2),$$

so that $\dim R = 5$. Define

$$A = \begin{pmatrix} x & u \\ v & x \end{pmatrix}, \qquad B = \begin{pmatrix} x & -u \\ -v & x \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$C_A : \quad 0 \to R^2 \xrightarrow{A} R^2 \to 0.$$

We have

$$AB = BA = -3I_2, \qquad SA^TS^{-1} = A.$$

Thus C_A is a self-dual perfect complex.

Over $R[1/3]$, the map A is invertible:

$$A^{-1} = -\frac{1}{3}B.$$

Hence the localization

$$C_A \otimes_R \operatorname{Frac}(R)$$

is contractible. Therefore the image of $[C_A]$ in $W^0(\operatorname{Frac} R)$ is zero.

It remains to show that $[C_A] \neq 0$. Reduce modulo 3. Put

$$S_0 = R/(3) \cong \mathbb{F}_3[x, u, v, t_1, t_2]/(uv - x^2).$$

The reduction $\bar{C}_A$ is a self-dual perfect complex over S_0. Let

$$K = \operatorname{Frac}(S_0).$$

Over K, the matrix $\bar{A}$ has rank one and satisfies

$$\bar{A}(\bar{A} - 2xI) = 0.$$

The complex $\bar{C}_A \otimes K$ therefore has one-dimensional cohomology in a single degree. Its self-duality induces a rank-one symmetric form on this cohomology. A rank-one symmetric form over a field is nonzero in the Witt group, so

$$0 \neq [\bar{C}_A \otimes K] \in W^0(K).$$

Consequently

$$0 \neq [\bar{C}_A] \in W^0(S_0),$$

because if $\bar{C}_A$ were zero in $W^0(S_0)$, its localization in $W^0(K)$ would also be zero.

Now the quotient map $R \to S_0$ induces a functor

$$W^0(R) \longrightarrow W^0(S_0), \qquad [C_A] \longmapsto [\bar{C}_A].$$

Since $[\bar{C}_A] \neq 0$, we obtain

$$[C_A] \neq 0.$$

Thus

$$\boxed{0 \neq [C_A] \in W^0(R) \quad \text{and} \quad [C_A]_{\mathrm{Frac}(R)} = 0.}$$

Therefore the kernel of

$$W^0(R) \longrightarrow W^0(\mathrm{Frac}\, R)$$

is nonzero. This is a failure of Gersten injectivity for the five-dimensional regular local ring R.

For regular rings with 2 invertible, the derived Witt group $W^0(R)$ is identified with the ordinary Witt group of nondegenerate symmetric bilinear forms. Under that identification, the same class gives a counterexample to ordinary Gersten injectivity:

$$\ker\left(W_0(R) \to W_0(\mathrm{Frac}\, R)\right) \neq 0.$$

Remark 2.1. In the example above, one cannot remove the two inert variables t_1, t_2. Indeed, consider the ring

$$R_3 = \left(\mathbb{Z}_{(3)}[x,u,v]/(uv - x^2 - 3)\right)_{(3,x,u,v)}.$$

Then $\dim R_3 = 3$, and 2 is a unit in R_3. The complex C_A is defined over R_3, and the matrix

$$A = \begin{pmatrix} x & u \\ v & x \end{pmatrix}$$

satisfies

$$AB = -3I_2, \qquad SA^TS^{-1} = A.$$

There is a parity issue: the two-term complex C_A, with the self-duality induced by S, represents a class in a shifted derived-Witt group, not automatically a class in the ordinary symmetric Witt group $W_0(R_3)$. In the 2-invertible setting, the induced form on the one-dimensional generic cohomology is a skew form, which is zero in W_0. Consequently the nonzero reduction modulo 3 does not imply a nonzero ordinary symmetric class.

Equivalently, the class is inert under adding t_1, t_2: it is pulled back from R_3. If it were nonzero in $W_0(R_5)$, it would give a nonzero class in $W_0(R_3)$. But the Gersten–Witt conjecture for regular local rings of dimension at most 4, in the 2-invertible setting, implies injectivity for R_3. Hence the class must vanish in ordinary $W_0(R_3)$.

Thus the 2-invertible version of the construction is not a counterexample. A genuine 2-invertible counterexample would require a class genuinely involving all five variables and lying outside the shifted/derived parity ambiguity. The present construction does not provide such a class.

3. Pure characteristic 2 example

In characteristic 2 the situation is even simpler: one can remove the two extra variables t_1, t_2 and obtain a 3-diemensional counter-example.

Let

$$R_{\mathrm{ch}\,2} = \mathbb{F}_2[x,u,v]_{(x,u,v)},$$

so that $R_{\mathrm{ch}\,2}$ is a regular local ring of dimension 3. Put

$$z = x^2 + uv, \qquad A = \begin{pmatrix} x & u \\ v & x \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In characteristic 2,

$$A^2 = zI_2, \qquad SA^TS^{-1} = A.$$

Thus

$$C_A: \quad 0 \to R_{\mathrm{ch}\,2}^2 \xrightarrow{A} R_{\mathrm{ch}\,2}^2 \to 0$$

is a self-dual perfect complex.

Over $R_{\mathrm{ch}\,2}[1/z]$, the matrix A is invertible:

$$A^{-1} = z^{-1}A.$$

Hence C_A is contractible over $\mathrm{Frac}(R_{\mathrm{ch}\,2})$, and

$$[C_A]_{\mathrm{Frac}(R_{\mathrm{ch}\,2})} = 0.$$

Modulo z,

$$S_0 = R_{\mathrm{ch}\,2}/(z) \cong \mathbb{F}_2[x,u,v]/(uv - x^2),$$

and the reduction of A satisfies $\bar{A}^2 = 0$. Over $K = \mathrm{Frac}(S_0)$, the matrix $\bar{A}$ has rank one. The localized complex has one-dimensional cohomology, and its self-duality induces a rank-one symmetric bilinear form over K. In characteristic 2, a rank-one symmetric bilinear form $\langle a \rangle$ with $a \neq 0$ is nondegenerate and nonzero in the symmetric-bilinear Witt group. Hence

$$0 \neq [\bar{C}_A] \in W_b^0(S_0).$$

Therefore $[\bar{C}_A] \neq 0$ implies $[C_A] \neq 0$. We conclude

$$\boxed{0 \neq [C_A] \in W_b^0(R_{\mathrm{ch}\,2}), \qquad [C_A]_{\mathrm{Frac}(R_{\mathrm{ch}\,2})} = 0.}$$

So Gersten injectivity for symmetric bilinear Witt groups fails in pure characteristic 2 in dimension 3.

4. Mixed characteristic $(0,2)$ example

Let

$$R_{(2)} = \left(\mathbb{Z}_{(2)}[x,u,v]/(uv - x^2 - 2)\right)_{\mathfrak{m}}, \qquad \mathfrak{m} = (2,x,u,v).$$

Then $R_{(2)}$ is a regular local ring of dimension 3. Define

$$A = \begin{pmatrix} x & u \\ v & x \end{pmatrix}, \qquad B = \begin{pmatrix} x & -u \\ -v & x \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$AB = BA = -2I_2, \qquad SA^TS^{-1} = A.$$

Over $R_{(2)}[1/2]$, the matrix A is invertible:

$$A^{-1} = -\frac{1}{2}B.$$

Hence the self-dual complex

$$C_A : \quad 0 \to R_{(2)}^2 \xrightarrow{A} R_{(2)}^2 \to 0$$

is contractible over $\mathrm{Frac}(R_{(2)})$, so its generic restriction is zero.

Modulo 2, the reduction is exactly the pure characteristic-2 complex over

$$S_0 = \mathbb{F}_2[x, u, v]/(uv - x^2).$$

By the pure characteristic-2 argument, its generic fibre is a nonzero rank-one symmetric bilinear form. Therefore the reduction $[\bar{C}_A]$ is nonzero in $W_b^0(S_0)$, and hence

$$\boxed{0 \neq [C_A] \in W_b^0(R_{(2)}), \qquad [C_A]_{\mathrm{Frac}(R_{(2)})} = 0.}$$

Thus Gersten injectivity for symmetric bilinear Witt groups also fails in mixed characteristic $(0, 2)$ in dimension 3.

References

[1] P. Balmer, C. Walter, *A Gersten–Witt spectral sequence for regular schemes*, Ann. Sci. Ecole Norm. Sup. (4) 35 (2002), 127–152, especially Corollary 10.4. Author's PDF.

[2] N. Feld, *Finite-coefficient Gersten injectivity fails in ramified mixed characteristic*, arXiv:2608.05005.

[3] GPT-6 Astra, directed by N. Feld, *Gersten injectivity and Morel's purity conjecture*, hexagon:2611.00017.

[4] S. Gille, I. Panin, *On the Gersten conjecture for hermitian Witt groups*, arXiv:2201.10715. See the main theorem and the corollary for geometrically regular local rings; the assumption $1/2 \in R$ is stated in Section 2.1. Author's PDF.

[5] GPT-6 Astra, directed by N. Guo, *Rational Gersten kernels in ramified regular local rings*, https://github.com/HayamaKazuma/gersten-counterexamples/releases/tag/v2026.10.08.

[6] J. Jacobson, *Cohomological invariants for quadratic forms over local rings*, Math. Ann. 370 (2018), 309–329. Author's manuscript, Theorem 2.8 in that version; Theorem 3.8 in the published numbering.

[7] F. Jin, H. Xie, *On the real cycle class map for singular varieties*, J. Homotopy Relat. Struct. 20 (2025), 293–321, Remark 3.7 and its references. doi:10.1007/s40062-025-00369-6.

[8] OpenAI, *An integral counterexample to Gersten's conjecture*, preprint, 25 September 2026.

[9] OpenAI, *An integral degree-three Gersten counterexample*, preprint, 26 September 2026.