Recall - $F : I \to \mathcal{C}$ (small cat; $\mathcal{C}$ = cat) $\quad \operatorname{colim}_I F$, if exists, is s.t.

$$\forall c \in \mathcal{C}, \quad \operatorname{Hom}(\operatorname{colim}_I F, c) = \operatorname{Hom}_{\operatorname{Fun}(I^{op}, Sets)}(*, \operatorname{Hom}_{\mathcal{C}}(F(-), c))$$

$\rightsquigarrow \mathcal{C}$ has I-colimits

$$\operatorname{Hom}(c, \lim_I F) = \operatorname{Hom}_{\operatorname{Fun}(I^{op}, Sets)}(*, \operatorname{Hom}_{\mathcal{C}}(c, F(-)))$$

I-limits

Ex $\mathcal{C} = Sets$ has ~~all~~ small colimits & limits:

$$\operatorname{colim} F = \coprod_{i \in I} F(i) \Big/ R \qquad R = \{(x,y) \in F(i) \times F(j) \mid \exists \phi : i \to j \in I, F(\phi)(x) = y\}.$$

$\mathcal{C} = Ab, \ldots$

- I is filtered if
  - $I \neq \emptyset$.
  - $\forall\ i, j \in I$, $\exists\ i \to k,\ j \to k$
  - $\forall\ i \rightrightarrows j$, $\exists\ i \rightrightarrows j \to k$ s.t. the 2 maps are =.

finite if $Ob(I)$, $Mor(I)$ are finite sets.

- A functor commutes w. I-colimits if $\forall\ I \xrightarrow{F} \mathcal{C}$,

$\Phi : \mathcal{C} \to \mathcal{D}$

$$\operatorname{colim}_I \phi(F) \xrightarrow{\sim} \phi(\operatorname{colim}_I F)$$

(limits) $\qquad \phi(\lim_I F) \xrightarrow{\sim} \lim_I \phi(F)$

- F is left exact if it commutes w. finite limits

  right exact — finite colimits

  exact = L & R exact.

Th In $\mathcal{C} = Set$, filtered colimits commute with finite limits.

$F : I \times J \to Set \qquad \operatorname{colim}_I \lim_J F \xrightarrow{\sim} \lim_J \operatorname{colim}_I F.$

Lemma: I filtered, J finite $\Rightarrow \forall\ G : J \to I$ has a cocone, i.e.

$$\begin{tikzcd} J \arrow[r] \arrow[dr, "G"'] & * \arrow[d, "e"] \\ & I \end{tikzcd} \qquad \exists\ G \Rightarrow e.$$

const. functor

Pf of Th $x \in LHS \iff \exists i \in I,\ x = (x_j)_{j\in J} \in F(i,j)_{j\in J}$.

define $\lambda(x) = (x_j)_{j\in J} \in \overset{RHS}{\cancel{\lim_J \operatorname{colim}_I F}}$. $=$

Show : λ is bijective.

Inj : If $(x_j)_{j\in J} \in F(i,j)$, $(y_j)_{j\in J} \in F(i',j)$

s.t. $\forall j,\ [x_j] = [y_j]$

$\Rightarrow \exists\ f_j : i \to i_j$, $g_j : i' \to i_j$ s.t. $F(f_j, 1_j)(x_j) = F(g_j, 1_j)(y_j)$

Apply lemma to the diagram of all f_j & g_j's

$\Rightarrow \exists\ f : i \to i''$, $g : i' \to i''$ s.t. $\forall j,\ F(f, 1_j)(x_j) = F(g, 1_j)(y_j)$

$$\Rightarrow \lim_J F(f, 1_j)\big((x_j)_{j\in J}\big) = \lim_J F(g, 1_j)\big((y_j)_{j\in J}\big)$$

$$\Rightarrow [(x_j)_{j\in J}] = [(y_j)_{j\in J}].$$

Surjective : similar with lemma. $\square$.

Prop forget : Ab $\to$ Set preserves & reflects filtered colimits.

Cor In Ab (Set), filtered colimits are exact.

Recall : τ topology $=$ $\left|\begin{array}{l} \mathcal{C}_\tau \in \mathrm{Cat} \\ \text{covering families in } \mathcal{C}_\tau. \end{array}\right.$

Top $=$ (2-)Cat. of topologies Ob(Top) $= \{\tau\}$.

$\mathrm{Hom}(\tau,\tau') = \{ F : \mathcal{C}_\tau \to \mathcal{C}_{\tau'} \mid \forall (U_\alpha \xrightarrow{f_\alpha} U) \in \mathrm{Cov}_\tau,\ \forall V \to U \in \mathcal{C}_\tau,$

- $(F(U_\alpha) \to F(U)) \in \mathrm{Cov}_{\tau'}$,
- $F(U_\alpha \times_U V) \simeq F(U_\alpha) \times_{F(U)} F(V)$. $\}$

Def $\mathcal{C} \in Cat$. A pseudo-functor (lax 2-functor) $\Phi : \mathcal{C} \to Cat$ is:

1) $\forall\, U \in \mathcal{C} \rightsquigarrow$ $\exists$ category $\Phi U \in Cat$

2) $\forall\, f : U \to V \in \mathcal{C} \rightsquigarrow \exists$ functor $\Phi f = f^* : \Phi V \to \Phi U$

3) $\forall\, U \in \mathcal{C}$, $\exists$ iso of functors $\varepsilon_U : id_U^* \simeq id_{\Phi U} : \Phi U \to \Phi U$

4) $\forall\, U \xrightarrow{f} V \xrightarrow{g} W \in \mathcal{C}$, $\exists$ iso $\alpha_{f,g} : f^* g^* \simeq (gf)^* : \Phi W \to \Phi U$

s.t. a) $\forall\, U \xrightarrow{f} V \in \mathcal{C}$, $\forall\, \eta \in \Phi V$,

$$\alpha_{id_U, f}(\eta) = \varepsilon_U(f^*\eta) : id_U^* f^* \eta \to f^* \eta$$

$$\alpha_{f, id_V}(\eta) = f^* \varepsilon_V(\eta) : f^* id_V^* \eta \to f^* \eta .$$

b) $\forall\, U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} T \in \mathcal{C}$, $\forall\, \theta \in \Phi(T)$,

$$\begin{tikzcd}
f^* g^* h^* \theta \arrow[r, "\alpha_{f,g}(h^*\theta)"] \arrow[d, "f^* \alpha_{g,h}(\theta)"'] & (gf)^* h^* \theta \arrow[d, "\alpha_{gf,h}(\theta)"] \\
f^* (hg)^* \theta \arrow[r, "\alpha_{f,hg}(\theta)"] & (hgf)^* \theta
\end{tikzcd}$$

Ex $\mathcal{C} = Sch$ $\quad \Phi(X) = PSh(X, Ab)$

Rk If $\begin{cases} \varepsilon_U = Id \\ \alpha_{f,g} = Id \end{cases}$ $\quad \Phi$ is a functor $\mathcal{C} \to Cat$, $\quad f^* g^* = (gf)^*$

- A morphism $F : \Phi \to \Psi$ is
  - $\forall\, U \in \mathcal{C}$, $\exists$ functor $FU : \Phi U \to \Psi U$
  - $\forall\, U \xrightarrow{f} V \in \mathcal{C}$, $\exists$ iso $\beta_f : f^* \circ FU \xrightarrow{\sim} FV \circ f^*$

s.t. $\forall\, U \xrightarrow{f} V \xrightarrow{g} W \in \mathcal{C}$,

$$\begin{array}{ccc} (gf)^* \circ FU & \simeq & FW \circ (gf)^* \\ \wr\wr & & \wr\wr \\ f^* \circ g^* \circ FU \simeq f^* \circ FV \circ g^* & \simeq & FW \circ f^* \circ g^* \end{array}$$

Fibered categories

Def 1) $\mathrm{Cat}/\mathcal{C} = \{\text{categories over } \mathcal{C}\}$

$= \{\mathcal{F} \in \mathrm{Cat} + p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}\}$.

For $X \in \mathcal{C}$, the fiber $\mathcal{F}(X)$ of $\mathcal{F}$ over X = subcat of $\mathcal{F}$

Obj: $\{Y \in \mathcal{F} \mid p_{\mathcal{F}}(Y) = X\}$

$\mathrm{Hom}(Y, Y') = \{\phi : Y \to Y'$ s.t. $p_{\mathcal{F}}(\phi) = id_X\}$

2) Let $\mathcal{F} \in \mathrm{Cat}/\mathcal{C}$

A morphism / arrow $\phi : A \to B$ in $\mathcal{F}$ is Cartesian if $\forall\, \psi : C \to B$ in $\mathcal{F}$

$\forall\, h : p_{\mathcal{F}} C \to p_{\mathcal{F}} A$ in $\mathcal{C}$

s.t. $p_{\mathcal{F}} \phi \circ h = p_{\mathcal{F}} \psi$, $\exists !\, \theta : C \to A$ s.t. $\left|\begin{array}{l} p_{\mathcal{F}} \theta = h \\ \phi \circ \theta = \psi \end{array}\right.$

$$\begin{tikzcd} C \arrow[r, dashed, "\exists \theta"] \arrow[rr, bend left, "\psi"] \arrow[d] & A \arrow[r, "\phi"] \arrow[d, mapsto] & B \arrow[d, mapsto] \\ p_{\mathcal{F}} C \arrow[r, "h"'] \arrow[rr, bend left] & p_{\mathcal{F}} A \arrow[r] & p_{\mathcal{F}} B \end{tikzcd}$$

We say that A is a pullback of B to $p_{\mathcal{F}} A$

Rk A pullback is unique up to unique isomorphism.

Def - A fibered category over $\mathcal{C}$ is a category $\mathcal{F} \in \mathrm{Cat}/\mathcal{C}$

s.t. $\forall\, U \xrightarrow{f} V \in \mathcal{C}$ $\cancel{p_{\mathcal{F}} A}$

$\forall\, B \in \mathcal{F}(V)$ s.t. $\cancel{p_{\mathcal{F}} B = V}$, $\exists$ Cart. morphism $\phi : A \to B$

s.t. $p_{\mathcal{F}} \phi = f$.

(i.e. pullback of any object of $\mathcal{F}$ along any morphism in $\mathcal{C}$ exists)

- A morphism of fibered categories is a functor $F : \mathcal{F} \to \mathcal{G}$

s.t. - $p_{\mathcal{G}} \circ F = p_{\mathcal{F}}$ (F preserves the base)

- F sends Cartesian morphisms to Cartesian morphisms.

Def $\mathcal{F} \to \mathcal{C}$ fibered cat. A <u>cleavage</u> is a class of Cartesian morphisms K in $\mathcal{F}$
$$\mathcal{F} \xrightarrow{p_{\mathcal{F}}} \mathcal{C}$$
s.t. $\forall\, U \xrightarrow{f} V \in \mathcal{C}$, $\forall\, B \in \mathcal{F}(V)$ $\quad \exists!\ \phi : A \to B \in K$

By AC, any f.c. has a cleavage.

Th $\left\{ \begin{array}{l} \text{fibered categories over } \mathcal{C} \\ \text{with a cleavage} \end{array} \right\} \overset{1 \text{ to } 1}{\longleftrightarrow} \{ \text{pseudofunctors on } \mathcal{C} \}$

(Grothendieck construction.)

P/ $\Rightarrow$):

$\mathcal{F} \to \mathcal{C}$ fibered. $U \in \mathcal{C}$, $\Phi U = \mathcal{F}(U)$.

$- \; U \xrightarrow{f} V \in \mathcal{C}$
$$\begin{tikzcd} K_Y \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$
$f^* : \mathcal{F}(V) \to \mathcal{F}(U)$
$Y \mapsto K_Y$

$- \; \varepsilon_U, \alpha_{f,g}$ follow from the universal property.

$\Leftarrow$: Φ pseudo-functor $\mathcal{C} \to \mathrm{Cat}$

$- \operatorname{Ob} \mathcal{F} = \{ (A, U) \mid U \in \mathcal{C},\ A \in \Phi U \}$.

$- \operatorname{Hom}((A,U), (B,V)) = \left\{ \begin{array}{l} f : U \to V \in \mathcal{C} \\ a : A \to \Phi f(B) \in \Phi U \end{array} \right\}$

$- \quad \mathcal{F} \to \mathcal{C}$
$\quad (A, U) \mapsto U$

$- \; \forall\, U \xrightarrow{f} V \in \mathcal{C}$, $\forall\, B \in \mathcal{F}(V)$,
$$\begin{array}{c} (\Phi f(B), U) \in \mathcal{F} \\ \big\downarrow (\mathrm{id}_{\Phi f(B)}, f) \\ (B, V) \end{array}$$
is a Cartesian morphism

$\Rightarrow$ cleavage. $\square$.

Def A cleavage is a <u>splitting</u> if it contains Id, is closed under composition.

Prop The pfunctor assoc. w. a cleavage is a <u>functor</u> $\Leftrightarrow$ it is a <u>splitting</u>

Ex – G group $\rightsquigarrow \mathcal{C}_G$ $\quad Ob = \{*\}$ $\quad$ (<u>groupoid</u>)

$\mathrm{Hom}(*, *) = G$.

– $G \to H$ grp hom $=$ functor $\mathcal{C}_G \to \mathcal{C}_H$

<u>Every</u> morphism is $\mathcal{C}_G$ is Cartesian

$\Rightarrow$ $\mathcal{C}_G \to \mathcal{C}_H$ fibered $\Leftrightarrow$ $G \twoheadrightarrow H$ surj.

- A cleavage $=$ $K \subset G$ subset, $K \leftrightarrow H$ bijection

splitting $\Leftrightarrow$ $K \subset G$ subgroup

$\Leftrightarrow$ a section $H \to G$ of $G \twoheadrightarrow H$

This does <u>not always</u> exist.

Th Every fibered cat. is <u>equivalent</u> to a split fib. cat.

Colimits in Cat & Top

$F : I \longrightarrow \mathrm{Cat}$ $\qquad \forall \mathcal{C} \in \mathrm{Cat}$,

$$\mathrm{Fun}(\operatorname{colim}_I F, \mathcal{C}) = \mathrm{Hom}_{\mathrm{Fun}(I^{op}, \mathrm{Cat})}(*, \mathrm{Fun}(F(-), \mathcal{C}))$$

Prop colim in Cat exist

D/ (I filtered). $\mathrm{Ob}(\operatorname{colim}_I F) = \coprod_{i \in I} \mathrm{Ob}\, F(i)$

- For $U \in \mathrm{Ob}\, F(i)$, $V \in \mathrm{Ob}\, F(j)$

Let $(i,j \backslash I) =$ Ob : $\left(\begin{smallmatrix} i \\ j \end{smallmatrix} \rightrightarrows k \right)$

Mor : $\begin{tikzcd} i \arrow[r] \arrow[rr, bend left] & k \arrow[r] & k' \\ j \arrow[ur] \arrow[urr] \end{tikzcd}$

non-empty, filtered

Let $H(U,V) : (i,j \backslash I) \longrightarrow \mathrm{Sets}$

$$\left(\begin{tikzcd} i \arrow[dr, "f"] & \\ & k \\ j \arrow[ur, "g"'] & \end{tikzcd} \right) \longmapsto \mathrm{Hom}(Ff(U), Fg(V))$$

Set $\mathrm{Hom}_{\operatorname{colim}_I F}(U,V) = \operatorname{colim}_{(i,j\backslash I)} H(U,V)$ $\square$.

<u>Recall</u> τ topology $= \left| \begin{array}{l} \mathcal{C}_\tau \in \mathrm{Cat} \\ \text{covering families in } \mathcal{C}_\tau. \end{array} \right.$

Top = <u>2-Cat. of topologies</u>

$\mathrm{Ob}(\mathrm{Top}) = \{\tau\}$

$$\mathrm{Hom}(\tau, \tau') = \left\{ F \in \mathrm{Fun}(\mathcal{C}_\tau, \mathcal{C}_{\tau'}) \;\middle|\; \begin{array}{l} \forall (U_\alpha \xrightarrow{f_\alpha} U) \in \mathrm{Cov}_\tau \\ \forall V \to U \in \mathcal{C}_\tau \\ -(F(U_\alpha) \to F(U)) \in \mathrm{Cov}_{\tau'} \\ -F(U_\alpha \times_U V) \simeq F(U_\alpha) \times_{F(U)} F(V) \end{array} \right\}$$

- pseudofunctors in Top $\qquad$ (F ~~commutes w~~. fiber products / preserves covering families)

Prop I filtered $T : I \to \mathrm{Top}$ pseudofunctor s.t. $\forall i \in I$, all covering families in $T(i)$ are finite. Then $\operatorname{colim}_I T$ is representable.

Pf/ Let $\mathcal{C}_{\mathcal{T}} = \operatorname{colim}_I \mathcal{C}_T \in \mathrm{Cat}$

- A finite family $\{U_\alpha \xrightarrow{\phi_\alpha} U\} \in \mathcal{C}$ is in $\mathrm{Cov}(\operatorname{colim}_I T)$

$$\left|\begin{array}{l} U \in \mathcal{C}_{T(i)} \\ U_\alpha \in \mathcal{C}_{T(i_\alpha)} \end{array}\right. \quad \text{if } \exists \left|\begin{array}{l} j \in I \\ i \xrightarrow{g} j \in I \\ i_\alpha \xrightarrow{g_\alpha} j \\ \phi_{\alpha_j} : g_\alpha(U_\alpha) \to g(U) \text{ representing } \phi \end{array}\right. \quad \text{s.t. } \{g_\alpha(U_\alpha) \xrightarrow{\phi_{\alpha_j}} g(U)\} \in \mathrm{Cov}\,T(j)$$

□

Notation - $F : \mathcal{C} \to \mathcal{D} \in \mathrm{Cat}$, $Y \in \mathcal{D}$

$\rightsquigarrow$ $I^F_Y \in \mathrm{Cat}$ $\quad \mathrm{Ob}\, I^F_Y = \{(X, \phi) \mid X \in \mathcal{C},\ \phi \in \mathrm{Hom}_{\mathcal{D}}(Y, F(X))\}$

$$\mathrm{Hom}((X_1, \phi_1), (X_2, \phi_2)) = \{ f \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \mid F(f) \circ \phi_1 = \phi_2 \}$$

- $\underset{FCF}{T : I \to \mathrm{Top}}$, $i \in I$, $U \in \mathcal{C}_{T(i)}$, $o \in I$

Define $J_U : o, i \backslash I \to \mathrm{Cat}$

$$\left(\begin{array}{c} o \xrightarrow{f} j \\ i \xrightarrow{g} j \end{array} \right) \longmapsto I^{T(f)}_{T(g)(U)}$$

Then $\operatorname{colim}_{o,i\backslash I} J_U \simeq J^{L_o}_U$, where $L_o : T(o) \to \operatorname{colim}_I T$.

(as functors)

Recall $F : \mathcal{C} \to \mathcal{D} \in \mathrm{Top}$ $\qquad \mathcal{F}_o \in \mathrm{PSh}(T(o))$

$F_\# : \mathrm{PSh}(\mathcal{C}) \rightleftarrows \mathrm{PSh}(\mathcal{D}) : F^\#$ $\qquad \Rightarrow (L_o)_\# \mathcal{F}_o \in \mathrm{PSh}(\operatorname{colim} T)$

$F^\# \mathcal{F} = \mathcal{F} \circ F$

$$F_\# \mathcal{F}(U) = \operatorname*{colim}_{(X,\phi) \in J^F_U} \mathcal{F}(X)$$

$F_* : \mathrm{Sh}(\mathcal{C}) \rightleftarrows \mathrm{Sh}(\mathcal{D}) : F^*$ $\qquad F^* \mathcal{F} = F^\# \mathcal{F}$

$\qquad F_* \mathcal{F} = a_{\mathcal{C}}(F_\# \mathcal{F})$

Assume: - $T_0 \in \mathrm{Top}$ FCF

- $I_0 = I \setminus \{0\}$ $\quad V : I_0^{op} \to \mathcal{C}_{\mathcal{T}}$

$i \mapsto V(i)$

- 0 is initial in I $\Rightarrow 0, i \setminus I \simeq i \setminus I$

- $\forall i, j \in I_0$, $\quad V_i \times_{V_j} U$ exist

$\forall U \in \mathcal{C}_{T_0}$ $\quad V_i \times U$

$T(0) = T_0$ $\quad T(i \xrightarrow{f} j) : T(i) \to T(j)$

- $T : I_0 \to \mathrm{Top}$ is s.t. $T(i) = \mathcal{T}/V_i$ $\quad U/V_i \mapsto U \times_{V_i} V_j / V_j$

$T(0 \xrightarrow{g} i) : T(0) \to T(i)$

i.e. - $\mathcal{C}_{T(i)} = \mathcal{C}_{\mathcal{T}}/V_i$ $\quad U \mapsto U \times V_i$

- $\mathrm{Cov}(\mathcal{C}_{T(i)}) =$ families $\{U_\alpha \to U\}$ over V_i s.t. $\{U_\alpha \to U\} \in \mathrm{Cov}\,\mathcal{T}$

$\Rightarrow$ - All $T(i)$ have FCF

- $\mathcal{T} = \operatorname{colim}_{I} T \in \mathrm{Top}$ exists $\quad L_0 : T_0 \to \mathcal{T}$

- For $F \in PSh(T_0, \cancel{Ab})$, Let $\underline{F} = (L_0)_\# F \in PSh(\mathcal{T})$

for $0 \xrightarrow{g} i \in I$, $F_i = (Tg)_\# F \in PSh(T(i))$

Th 1) Let $U \in \mathcal{C}_{\mathcal{T}}$, with $U \in \mathcal{C}_{T(i)}$

$F \in T_0$

Then $\underline{F}(U) \simeq \operatorname{colim}_{i \xrightarrow{g} j \in i \setminus I} F_i(T(g)(U)) \simeq \operatorname{colim}_{i \to j \in i \setminus I} F(U \times_{V_i} V_j)$

2) ~~If~~ $F \in Sh(T_0)$ ~~then~~ $\Rightarrow$ $\underline{F} \in Sh(\mathcal{T})$

F flasque $\Rightarrow \underline{F}$ flasque

3) $(L_0)_\#$ & $(L_0)_* = (L_0)_\# |Sh(T_0)$ are exact.

4) ~~If~~ $\forall F \in Sh(T_0)$, $\forall U \in \mathcal{C}_{T_0}$,

$H^p(\mathcal{T}, U; \underline{F}) \simeq \operatorname{colim}_{i \in I} H^p(T_j, U \times V_j; F_j) \simeq \operatorname{colim} H^p(T_0, U \times V_j; F)$

D/ i): colim commute w/ colim

ii) filtered colim commute with $\oplus$ & finite lim.

iii) Same

iv) universality of HP. $\square$.

Th (EGA IV 8.8.2) I filtered $X : I^{op} \to Sch$, $i \mapsto X(i)$

$Sch_{X(i)} = \{$ separated $X(i)$-schemes FP $\}$ $\Rightarrow S : I \to Cat$ p-functor, $i \mapsto Sch_{X(i)}$

Assum: - $\forall i \in I$, $X(i)$ is qcqs

- $\forall i \to j \in I$, $X(j) \to X(i)$ is an affine morphism

(inverse image of any affine is affine)

Then: - $\underline{X} = \lim_I X$ exists $\in Sch$

- $\operatorname{colim}_I S \xrightarrow{\sim} Sch_{\underline{X}}$ eq. of cat.

- $0 \in I$ initial [redacted] T_0 = top. on X_0 FCF

As before: - T_i = top. on $X(i)$ (induced by T_0)

- $T_{\underline{X}}$ = top. on $\underline{X}$

$\Rightarrow$ - $T : I \to Top$ p-functor

Case (f): etale topology
sep FP
finite surjective fa.

~~$\operatorname{colim}_I T \to T_{\underline{X}}$~~

Th $\operatorname{colim}_I T \to T_{\underline{X}}$ is an eq. in Top.

D/ $\mathcal{C}_{T_i} \subset S(i)$, $\mathcal{C}_{T_{\underline{X}}} \subset Sch_{\underline{X}}$ full subcat $\Rightarrow \mathcal{C}_{\operatorname{colim} T} \subset \operatorname{colim}_I S \simeq Sch_{\underline{X}}$ full subcat

$\Rightarrow \mathcal{C}_{\operatorname{colim} T} \to \mathcal{C}_{T_{\underline{X}}}$ fully faithful.

Need to check :

a) $\forall\, U \in \mathcal{C}_{T_{\underline{X}}}$, $\exists \left| \begin{array}{l} i \in I \\ U_i \in \mathcal{C}_{T_i} \end{array} \right.$ $\quad U \simeq U_j \times_{X_j} \underline{X}$

b) $\forall \{U \to V\} \in \mathrm{Cov}\, T_{\underline{X}}$, $\exists \left| \begin{array}{l} i \in I \\ \{U_i \to V_i\} \in \mathrm{Cov}\, T_i \end{array} \right.$ $\quad U \simeq U_j \times_{X_j} \underline{X}$, $\quad V \simeq V_j \times_{X_j} \underline{X}$

($\{U_\alpha \to V\}$ finite cover $\Leftrightarrow$ $\{\amalg U_\alpha \to V\}$)

a) : WMA $\left. \begin{array}{l} X_i = \mathrm{Spec}\, A_i \\ \underline{X} = \mathrm{Spec}\, \underline{A} \\ U = \mathrm{Spec}\, B \end{array} \right)$ $\Omega^1_{B/\underline{A}} = 0$ $\Rightarrow \exists \left| \begin{array}{l} i \in I \\ B_i/A_i \text{ FP} \end{array} \right.$ s.t. $B_i \otimes_{A_i} \underline{A} \simeq B$

For $i \to j$, set $B_j = B_i \otimes_{A_i} A_j$ $\Rightarrow \exists j$, B_j/A_j flat (EGA IV 11.2.6.1)

$\Rightarrow \exists K$, $\Omega^1_{B_K/A_K} = 0$ $\Rightarrow B_K/A_K$ etale.

b) $U \to V \in \mathcal{C}_{T_{\underline{X}}}$ surjective

By (a), $\exists \left| \begin{array}{l} i \in I, \\ U_i \to V_i \in \mathcal{C}_{T(i)} \end{array} \right.$ $\quad (U \to V) \simeq (U_i \to V_i) \times_{X(i)} \underline{X}$

$\uparrow$
etale FP

$\Rightarrow \widetilde{V}_i = \mathrm{Im}(U_i) \subset V_i$ open, $\widetilde{V}_i \to V_i$ FP

$\Rightarrow U_i \to \widetilde{V}_i \in \mathrm{Cov}\, T(i)$

$\Rightarrow$ $V_i \times_{X_i} \underline{X} \simeq V$ $\square$.

Cor $X_0 \in \mathrm{Sch}$, $X : I^{op} \to \mathcal{C}_{T_{X_0}}$ in (f, 0, 1, 2) s.t. $\left| \begin{array}{l} \forall i,\ X(i) \text{ qcqs} \\ X(i) \leftarrow X(j) \text{ affine.} \end{array} \right.$ $\quad \underline{X} = \lim_{I^{op}} X$

$i \mapsto X(i)$

$\mathcal{F} \in \mathrm{Sh}(X_0)$ $\quad \mathcal{F}_i \in \mathrm{Sh}(X(i))$ $\quad \underline{\mathcal{F}} \in \mathrm{Sh}(\underline{X})$ $\qquad$ Even better: $\mathrm{colim}_I\, D(X(i)) \simeq D($

Then $\mathrm{colim}_I\, H^q(X(i), \mathcal{F}_i) \simeq H^q(\underline{X}, \underline{\mathcal{F}})$.