The Chow weight obstruction

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Abstract

We study the existence of Chow weight structure on the motivic stable homotopy category. If the base scheme has a proper and regular model with a formally real point, or if 2 is not invertible in the coefficients, we show its non-existence. With rational coefficients, we obtain a necessary and sufficient condition for its existence over schemes that are separated of finite type over a separated excellent base scheme of dimension at most 3.

Let S be a nonempty, noetherian and finite dimensional scheme. Write S_r for its real spectrum, whose points are pairs consisting of a point s and an ordering of $\kappa(s)$. Write $R \subseteq \mathbb{Q}$ for a nonzero localization of $\mathbb{Z}$ and $\mathrm{SH}(S)_R$ for the corresponding localization of the motivic stable homotopy category. Let $C(A) = A(1)[2]$ denote the Chow twist.

In the definition of Déglise–Fasel–Jin–Khan [DFJK, §8.12], a Chow weight heart contains $C^r(f_*\mathbf{1}_X)$ for every proper $f : X \to S$ with X regular and every $r \in \mathbb{Z}$. Our nonexistence results only use this containment. The orthogonality axiom for a weight structure forces $\mathrm{Hom}(A, B[n]) = 0$ for any two objects of its heart and every $n > 0$. Corollary 8.14 of [DFJK] obtains a contradiction using a regular proper model of a generically finite map in characteristic zero.

An obstruction over a proper source

Theorem 1. *Let the pair (S, R) be as above. Assume that one of the following conditions hold:*

1. *There exists a proper morphism $f : X \to S$ with X nonempty and regular, such that $X_r \neq \varnothing$.*
2. *$2 \notin R^\times$.*

Then no weight heart in $\mathrm{SH}(S)_R$ contains all the prescribed Chow objects.

Proof. Write $\eta_S : \mathbf{1}_S(1)[1] \to \mathbf{1}_S$ for the stable motivic Hopf map. For a perfect field F of any characteristic and $n > 0$, Morel's calculation of the diagonal stable stems gives

$$[\mathbf{1}_F(n)[n], \mathbf{1}_F] \simeq K^{MW}_{-n}(F) \simeq W(F).$$

Here $W(F)$ is the Witt ring of nonsingular symmetric bilinear forms, and η_F^n corresponds to $\langle 1 \rangle$ [RS, Thm. 3.10] [M, Lem. 3.10]. Compactness of the sphere and its suspensions identifies the localized group with $W(F) \otimes_{\mathbb{Z}} R$.

If $X_r \neq \varnothing$, take F to be a real closure at an ordered point of X. Signature sends $\langle 1 \rangle \otimes 1$ to $1 \in R$.

If $2 \notin R^\times$, choose a closed point y of the nonempty noetherian scheme S. The closed immersion $X = \operatorname{Spec} \kappa(y) \hookrightarrow S$ is proper with nonempty regular source, and take F to be an algebraic closure of $\kappa(y)$.

Every metabolic form has even rank, so rank modulo two induces a homomorphism $W(F) \otimes_{\mathbb{Z}} R \to R/2R$. It sends $\langle 1 \rangle \otimes 1$ to $1 \neq 0$, including when $\operatorname{char} F = 2$. In either case the pullback of η_X^n to F is nonzero.

Put $A = f_* \mathbf{1}_X$ in $\mathrm{SH}(S)_R$. We show that for every $n > 0$ the group $\operatorname{Hom}(C^n A, A[n])$ contains a nonzero morphism.

Tensoring η_S^n with A gives $\eta_A^n : A(n)[n] \to A$. For the adjunction unit $u : \mathbf{1}_S \to f_* \mathbf{1}_X$, its composite with $u(n)[n]$ is $u \circ \eta_S^n$. Under $f^* \dashv f_*$, this last map corresponds exactly to η_X^n, and is therefore nonzero. It follows that $\eta_A^n[n] : C^n A \to A[n]$ is nonzero. The two unshifted objects $C^n A$ and A are among the prescribed Chow objects, contradicting orthogonality. □

For example, the integral obstruction holds over $\operatorname{Spec} \mathbb{F}_2$. For any odd prime p, it also holds with $\mathbb{Z}[1/p]$-coefficients over every nonempty finite-dimensional noetherian $\mathbb{F}_p$-scheme.

Real points and rational Chow weights

Theorem 2. *Let S be separated and of finite type over a separated excellent noetherian scheme B of dimension at most three. Then*

$$\mathrm{SH}(S)_{\mathbb{Q}} \text{ admits the Chow weight structure} \quad \Longleftrightarrow \quad S_r = \varnothing.$$

Proof. Suppose $S_r \neq \varnothing$. Choose an ordered point of S with underlying point s and image $b \in B$. For this direction it suffices that B be quasi-excellent and that $\dim \overline{\{b\}} \leq 3$. Give $k = \kappa(b)$ the ordering induced by $\kappa(s)$. Choose an affine neighbourhood $\operatorname{Spec} A \subset S_b$ of s. The quotient of A by the prime of s is a finite-type k-domain with ordered fraction field. The Artin–Lang theorem supplies a homomorphism from this domain to the real closure k^{rc} for the chosen ordering [Sw, Thm. 4.15]. Its image, being a finitely generated algebraic k-algebra and a domain, is a finite field extension of k. We have found $y \in S_b$ with $\kappa(y)/k$ finite and formally real.

Let $Z = \overline{\{y\}}_{\mathrm{red}} \subset S$ and $D = \overline{\{b\}}_{\mathrm{red}} \subset B$. The morphism $Z \to D$ is dominant, of finite type and generically finite. The dimension inequality gives $\dim Z \leq \dim D \leq 3$ [SP, Tag 02JT]. As Z is integral, separated and quasi-excellent, resolution in dimension at most three, due in dimension three to Cossart–Piltant [CP, Thm. 1.1], gives a proper birational $Y \to Z$ with regular integral Y; see also [B, Thm. 2.13]. The composite $Y \to S$ is proper, and $K(Y) = \kappa(y)$ is formally real. Theorem 1 therefore rules out a weight heart containing the prescribed Chow objects for every nonzero localization R of $\mathbb{Z}$. Taking $R = \mathbb{Q}$ proves necessity.

Conversely, suppose $S_r = \varnothing$. Then the rational minus summand vanishes, and the equivalence $\mathrm{SH}(S)_{\mathbb{Q}} \simeq \mathrm{DM}_{\mathrm{B}}(S)$ with Beilinson motives identifies units, Tate twists, and proper pushforwards [DFJK, proof of Prop. 4.4].

In $\mathrm{DM}_{\mathrm{B}}(S)$, let $\mathcal{H}$ consist of the objects $p_* \mathbf{1}_T(r)[2r]$ for proper $p : T \to S$ with T regular and $r \in \mathbb{Z}$. These are compact and lie in the heart of the bounded Chow weight structure on constructible Beilinson motives [BS, Ex. 4.1.1(1) and Prop. 4.1.3]. In particular $\operatorname{Hom}(H, H'[i]) = 0$ for $H, H' \in \mathcal{H}$ and $i > 0$. Moreover the projective members of $\mathcal{H}$ already generate the compact category as a thick subcategory: apply [BD, §2.4.1, (Resol)(ii), and Cor. 2.4.8] to the rational, separated category of Beilinson motives, using $\dim B \leq 3$. The shifts $[2r]$ do not affect thick generation.

Thus $\mathcal{H}$ is a negative family of compact generators of the whole category. The weight-structure construction for such a family [Bon, Prop. 1.2.3 and proof of Thm. 2.1.1(II)] gives a heart consisting of retracts of coproducts of its members. This is the Chow heart specified in [DFJK, §8.12]. Transport along the equivalence gives the assertion for $\mathrm{SH}(S)_{\mathbb{Q}}$. □

The dimension bound concerns the base B. In particular, the criterion applies to separated schemes of finite type over $\operatorname{Spec}\mathbb{Z}$ of any dimension. In this case, if $S_r \neq \varnothing$, the witness Z is an arithmetic curve with an orderable number field as its function field. Its finite normalization already supplies the regular Y.

Over $\operatorname{Spec}\mathbb{C}$, Theorem 1 gives the integral obstruction and Theorem 2 gives rational existence.

Bounded weights on compact spectra

For general S, a variant removes the proper-source hypothesis at the cost of a condition on the weight structure. Suppose $S_r \neq \varnothing$. There is no bounded weight structure on $\mathrm{SH}(S)^c_R$ for which the Chow twist $C = (1)[2]$ is weight-exact. Indeed, if $\mathbf{1}_S$ has weights in $[a, b]$, then $C^n\mathbf{1}_S$ has weights at most b while $\mathbf{1}_S[n]$ has weights at least $a + n$. For $n > b - a$ orthogonality forces $\mathrm{Hom}(C^n\mathbf{1}_S, \mathbf{1}_S[n]) = 0$. But $\eta_S^n[n]$ is nonzero after pullback to a real closure at an ordered point. If $S \neq \varnothing$ and $2 \notin R^\times$, pullback to a geometric point detects η_S^n by rank modulo two, as in the proof of Theorem 1. The same contradiction follows. In this argument finite weight amplitude of the unit suffices. For a bounded Chow heart generated by the Tate-stable family of Chow objects, C preserves the heart and hence is weight-exact, by bounded generation of the two weight classes from the heart.

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