GROTHENDIECK TOPOLOGIES

Notes on a Seminar by

M. Artin

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HARVARD UNIVERSITY
Department of Mathematics

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CHAPTER I. Preliminaries

Section 0. Introduction.

EXAMPLE (0.0). Let X be a topological space and let T be the category whose objects are open sets of X and with Hom (U, V) consisting of the inclusion map if $U \subset V$, empty otherwise. Then a presheaf F on X with values in a category C is a contravariant functor $F : T^{0} \longrightarrow C$. A sheaf F is a presheaf satisfying the following axiom: For $U \in T$, $\{U_i\}$ an open covering of U the sequence

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

is exact with the canonical maps. (We assume C has products. For the definition of exactness, see (Sem. Bourb. # 195)). Note that in T (in fact in the category of topological spaces) $U_i \cap U_j \approx U_i \times_U U_j$. Therefore,

DEFINITION (0.1). A Grothendieck topology T consists of a category Cat T and a set Cov T of families $\{U_i \xrightarrow{\phi_i} U\}_{i \in I}$ of maps in Cat T called coverings (where in each covering the range U of the maps ϕ_i is fixed) satisfying

(1) If ϕ is an isomorphism then $\{\phi\} \in \text{Cov } T$.

(2) If $\{U_i \longrightarrow U\} \in \text{Cov } T$ and $\{V_{ij} \longrightarrow U_i\} \in \text{Cov } T$ for each i then the family $\{V_{ij} \longrightarrow U\}$ obtained by composition is in Cov T.

(3) If $\{U_i \longrightarrow U\} \in \text{Cov } T$ and $V \longrightarrow U \in \text{Cat } T$ is arbitrary then $U_i \times_U V$ exists and $\{U_i \times_U V \longrightarrow V\} \in \text{Cov } T$.

We will abuse language and call T a topology.

DEFINITION (0.2). Let T be a topology and C a category with products. A presheaf on T with values in C is a functor $F : T^0 \longrightarrow C$. A sheaf F is a presheaf satisfying

(S) If $\{U_i \longrightarrow U\} \in \text{Cov } T$ then the diagram

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is exact.

We shall restrict our attention to presheaves and sheaves with values in (Ab) or (Sets), (mostly (Ab)). Note that if F has values in (Ab) then F is a sheaf iff. it "is" a sheaf of sets.

EXAMPLE (0. 3). Let C be any category with fibered products. Then there is a canonical topology T_C associated to C, namely set Cat T_C = C, Cov T_C = set of families of maps $\{U_i \longrightarrow U\}$ in C which are universal effective epimorphisms, i.e., families satisfying $\forall Z \in Ob\, C$

$$\mathrm{Hom}(U, Z) \longrightarrow \prod \mathrm{Hom}(U_i, Z) \rightrightarrows \prod \mathrm{Hom}(U_i \underset{U}{\times} U_j, Z)$$

is exact, and similarly for a base extension $\{U_i \underset{U}{\times} V \longrightarrow V\}$. (One has to check the axioms).

TAUTOLOGY (0.4). Every representable functor on C is a sheaf of sets on T_C. It is useful to know that the coverings in a topology are universal effective epimorphisms, preferably in some large category, so that one can lay ones hands on some sheaves. For instance, in example (0. 0) the coverings are universal effective epimorphisms in the category of all topological spaces, i.e., $\mathrm{Hom}_{Top}(\ , Y)$ is a sheaf on T for Y a topological space.

EXAMPLE (0. 5). Let $T = T_{(Sets)}$. Then the coverings $\{U_i \longrightarrow U\}$ are families of maps which are surjective (i.e. such that U is covered by the union of the images of the U_i's). One verifies easily that every sheaf of sets on T is a representable functor, namely $F(U) \approx \mathrm{Hom}(U, F(e))$ where e is a set of one element. More generally

EXAMPLE (0. 6). Let G be a group and T_G the canonical topology on the category of left G-sets (sets with G operating). Again coverings are families of G-maps which are surjective, and every sheaf of sets is a representable functor, in fact $F(U) \approx \mathrm{Hom}_G(U, F(G))$ where $F(G)$ is obtained by viewing G as a left G-set and has as operation of $\sigma \in G$ the one induced by right multiplication on G by σ^{-1}.

Therefore the category of abelian sheaves on T_G is equivalent with the category of G-modules, and so the derived functors of a suitable left exact functor Γ : ab. sheaves $\longrightarrow$ (Ab) will be the ordinary cohomology of groups with values in the corresponding G-module. The functor Γ is of course $F \rightsquigarrow F(e)$ where e is a set of one element with its unique structure of G-set.

EXAMPLE (0. 6 bis). Let G be a profinite group and set Cat T_G = Category of finite sets with continuous G-operation; Cov T_G = finite families of maps which are surjective. One may verify that the category of abelian sheaves on T_G is equivalent with the category of continuous G-modules (although this is not the canonical topology on Cat T_G, and only the "finite" sheaves are representable). Hence, taking derived functors of $\Gamma : F \rightsquigarrow F(e)$ one obtains the Tate cohomology groups.

EXAMPLE (0. 7). Let X be a ~~noetherian scheme, and define~~ T_X by.

Cat T_X = category of schemes Y/X étale, finite type.

Cov T_X = finite surjective families of maps.

These coverings are universal effective epimorphisms in the category of all preschemes.

For instance, if X = spec k, k a field, then Cat T_X is dual to the category of finite separable (commutative) k-algebras and by Galois theory is equivalent to the category of finite sets with continuous $G(\bar{k}/k)$ operation, where $G(\bar{k}/k)$ is the Galois group of the separable algebraic closure $\bar{k}$ of k. As in (0. 6 bis) one gets Galois cohomology. This example will be examined in more detail later.

Section 1. Generalities on $\varinjlim$. Let I, C be categories and $F : I \longrightarrow C$ a functor. For $X \in C$ denote by $c_X : I \longrightarrow C$ the constant functor carrying Ob I to X and Fl I to id_X. We obtain a covariant functor $\mathrm{Hom}_{fun}(F, c_X) : C \longrightarrow$ (Sets) which is denoted by $\underrightarrow{\lim} F$. If this functor is representable the object representing it is called $\varinjlim F$. Dually, $\mathrm{Hom}(c_X, F)$ is denoted by $\underleftarrow{\lim} F$.

Let us write F_i for $F(i)$, ($i \in$ Ob I).

PROPOSITION (1. 1). If C = (Sets) (resp. (Ab)) then $\underrightarrow{\lim} F$ is representable.

Proof: Take

$$\varinjlim F = \coprod_{i \in I} F_i/R \qquad\qquad (\text{resp. } \bigoplus_{i \in I} F_i/R\ )$$

where R is the equivalence relation (resp. subgroup) generated by pairs (x, y) (resp. elements $x-y$), say $x \in F_i$, $y \in F_j$, such that $\exists\ \phi : i \longrightarrow j$ in I with $[F(\phi)](x) = y$.

Suppose C = (Sets) then R can be described as follows: For $x \in F_i$, $y \in F_{i'}$ we have $(x, y) \in R$ iff. $\exists\ i = i_0, i_1, \cdots, i_n = i'$ and $j_1, \cdots, j_n$ and $x_\nu \in F_{i_\nu}$ $(x_0 = x,\ x_n = y)$; $z_\nu \in F_{j_\nu}$ and a diagram

$$(*) \qquad i_0 \longrightarrow j_1 \longleftarrow i_1 \longrightarrow \cdots\ j_n \longleftarrow i_n \qquad \text{in } I$$

with

$$x = x_0 \rightsquigarrow z_1 \leftsquigarrow x_1 \rightsquigarrow \cdots \leftsquigarrow x_n = y$$

under the induced maps. A diagram (*) is called a connection of (i, i') in I. I is connected iff. every pair (i, i') of elements has a connection. Obviously if $I = \coprod_\alpha I_\alpha$ is a direct sum (i.e. disjoint union) of categories I then there is a canonical isomorphism

$$\coprod_{\alpha} \varinjlim F|I_{\alpha} \longrightarrow \varinjlim F$$

(this doesn't depend on C). In particular, if I is a <u>discrete</u> category (a category in which the arrows are reduced to identity maps) and if direct sums exist in C then

$$\varinjlim F \approx \coprod_{i \in I} F_i \quad .$$

The following axioms for I are useful:

(L1) Given a diagram

$$\begin{array}{ccc} & & j \\ & \nearrow & \\ i & & \\ & \searrow & \\ & & j' \end{array} \quad \text{in } I, \ \exists \quad \begin{array}{ccc} j & & \\ & \searrow & \\ & & k \\ & \nearrow & \\ j' & & \end{array}$$

such that the resulting square commutes.

(L2) Given a diagram $i \rightrightarrows j$, $\exists$ a map $j \longrightarrow k$ such that the two maps $i \longrightarrow k$ obtained by composition are the same.

(L3) I is connected.

Note that if C = (Sets) and (L1) holds then the equivalence relation R reads $x \sim y$ (say $x \in F_i$, $y \in F_j$) iff.

$$\exists \quad \begin{array}{ccc} i & & \\ & \searrow & \\ & & k \\ & \nearrow & \\ j & & \end{array} \quad \text{in } I$$

such that the induced images of x, y in F_k are equal. In other words, one may talk more or less as is usual with inductive systems if (L1) holds in I and C = (Sets).

Now let C = (Ab) and denote by $\mathcal{F}$ the category of functors $F : I \longrightarrow (Ab)$. $\mathcal{F}$ is an abelian category and

PROPOSITION (1. 2). The functor $\mathcal{F} \longrightarrow (Ab)$ defined by $F \rightsquigarrow \varinjlim F$ is right exact.

We omit the proof.

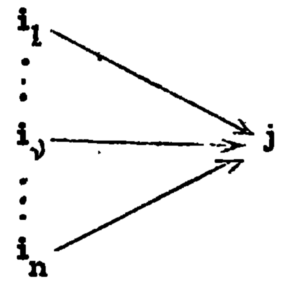
Let $F \in \mathcal{F}$ and denote by $\text{set} \varinjlim F$ the limit of the underlying sets of F. Then by definition of $\varinjlim$ for sets there is a map $\text{set} \varinjlim F \longrightarrow \varinjlim F$. In general this map is not bijective. However

PROPOSITION (1. 3). Suppose $I \neq \emptyset$ and that (L, 1, 2, 3) hold in I. Then $\text{set} \varinjlim F \longrightarrow \varinjlim F$ is bijective.

<u>Proof:</u> Suppose first (L1, 3) hold, and let

$$\sum_{\nu=1}^{n} x_\nu \in \bigoplus_{i \in I} F_i$$

where say $x_\nu \in F_{i_\nu}$. Applying (L3) and (L1) we can find a diagram



in I. Let $z_\nu \in F_j$ be the image of x_ν under the induced map, and set $z = \Sigma z_\nu$. Then $\Sigma x_\nu \equiv z \pmod{R}$ where R is the subgroup defined in the proof of (1.1). This shows that the map

$$\coprod_{i \in I} F_i \longrightarrow \varinjlim F$$

is surjective, and hence $\underrightarrow{\text{set lim}}\, F \longrightarrow \varinjlim F$ is obviously surjective.

Now suppose also (L2) holds. We shall define a group law on $\underrightarrow{\text{set lim}}\, F$ so that the maps $F_i \longrightarrow \underrightarrow{\text{set lim}}\, F$ are homomorphisms. This will induce a map $\varinjlim F \longrightarrow \underrightarrow{\text{set lim}}\, F$ which, composed with the one under consideration, gives the identity, proving the proposition.

Let $\bar{x}, \bar{y} \in \underrightarrow{\text{set lim}}\, F$ be represented by x, y. We may assume $x, y \in F_i$, $i \in I$. Set $\bar{x} + \bar{y} = \overline{x + y}$. The group axioms will be trivial if this is well defined, as will be the fact that $F_i \longrightarrow \underrightarrow{\text{set lim}}\, F$ is a homomorphism. So suppose $\bar{x}, \bar{y}$ are also represented by $x', y' \in F_{i'}$.

Then there exist diagrams

$$\begin{array}{ccc} i & & \\ & \searrow & \\ & & j \\ & \nearrow & \\ i' & & \end{array} \quad ; \quad \begin{array}{ccc} i & & \\ & \searrow & \\ & & j' \\ & \nearrow & \\ i' & & \end{array}$$

in I with $x = x'$ in F_j ; $y = y'$ in $F_{j'}$. Apply (L1) to the diagram

$$\begin{array}{ccc} & & j \\ & \nearrow & \\ i & & \\ & \searrow & \\ & & j' \end{array}$$

to get

$$\begin{array}{ccccc} & & j & & \\ & \nearrow & & \searrow & \\ i & & & & k \\ & \searrow & & \nearrow & \\ & & j' & & \end{array}$$

which commutes. We get two maps $i' \longrightarrow k$. Apply (L2) to find $k \longrightarrow k'$ such that the two maps $i' \longrightarrow k'$ are equal. Then

$$\begin{array}{ccccc} & & j & & \\ & \nearrow & & \searrow & \\ i & & & & k' \\ & \searrow & & \nearrow & \\ & & j' & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & j & & \\ & \nearrow & & \searrow & \\ i' & & & & k' \\ & \searrow & & \nearrow & \\ & & j' & & \end{array}$$

both commute and so the uniquely determined images in $F_{k'}$ of x, x' (resp. y, y') are equal. Hence $\overline{x + y} = \overline{x' + y'}$.

COROLLARY (1.4). If (L1, 2) hold in I then $\varinjlim$ is an exact functor : $\mathcal{J} \longrightarrow$ (Ab).

It is clear that $\underline{\text{set lim}}\ F' \hookrightarrow \underline{\text{set lim}}\ F$ if $F' \hookrightarrow F$ (i.e. $F'_i \hookrightarrow F_i$ for each i). Hence we are done if also (L3) holds by (1.2) and (1.3). If (L3) does not hold write $I = \coprod_\alpha I_\alpha$ with I_α connected and use the fact that $\oplus$ is an exact functor.

DEFINITION (1.5). A subcategory $J \subset I$ is a final subcategory (formerly : cofinal) iff.

(i) J is a full subcategory.

(ii) For all $i \in I$, $\exists$ $i \longrightarrow j$ in I with $j \in J$.

One verifies easily the following:

PROPOSITION (1.6). Let $J \subset I$ be a final subcategory. (Lx) for I $\Longrightarrow$ (Lx) for J (x = 1, 2, 3). If I satisfies (L1) and $F : I \longrightarrow C$ is arbitrary then the canonical map $\varinjlim F \longrightarrow \varinjlim F|J$ is bijective. In particular, if I has a final object ∞ (equivalently : $\{\infty, id_\infty\}$ is a final subcategory of I and (L1) holds in I) then $\varinjlim F \approx F_\infty$.

Let again I be given, and $a \in I$. Denote by $a\backslash I$ the category of maps $a \longrightarrow i$ of I and if $F : I \longrightarrow C$ is a functor, denote by $a\backslash F : a\backslash I \longrightarrow C$ the functor $a\backslash F\ (a \longrightarrow i) = F(i)$.

PROPOSITION (L 7). Let I, a, F be as above. Then (L1) for I $\Longrightarrow$ (L1, 3) for $a\backslash I$ and (L2) for I $\Longrightarrow$ (L2) for $a\backslash I$. If I satisfies (L1, 2, 3) then the canonical map $\varinjlim F \longrightarrow \varinjlim a\backslash F$ is bijective.

We will verify the last assertion : Let $X \in C$. We have to show that the obvious map $\mathrm{Hom}(F, c_X) \longrightarrow \mathrm{Hom}(a\backslash F, a\backslash c_X)$ is bijective. This means we have to show that an element $\xi \in \mathrm{Hom}(a\backslash F, a\backslash c_X)$ (i. e. a collection of maps $F_i \longrightarrow X$ for each $a \longrightarrow i \in a\backslash I$ with the appropriate commuting relations) determines a unique element of $\mathrm{Hom}(F, c_X)$. So let ξ be given and try to define for $i \in I$ a map $F_i \longrightarrow X$. Applying (L3), we find a diagram

$$\begin{array}{ccl} i & \searrow & \\ & & j \qquad \text{in } I \\ a & \nearrow & \end{array}$$

Hence ξ includes a map $F_j \longrightarrow X$ and we are forced to define $F_i \longrightarrow X$ by composition $F_i \longrightarrow F_j \longrightarrow X$. The commuting relations will be obvious if this is well defined. Suppose also

$$\begin{array}{ccl} i & \searrow & \\ & & i' \qquad \text{in } I \quad . \\ a & \nearrow & \end{array}$$

Apply (L1) to the diagram $\begin{array}{ccl} & \nearrow & j \\ i & \longrightarrow & j' \end{array}$ to find

$$\begin{array}{ccccc} & & j & & \\ & \nearrow & & \searrow & \\ i & & & & k \\ & \searrow & & \nearrow & \\ & & j' & & \end{array}$$

which commutes. We get two maps $a \longrightarrow k$, and applying (L2) we may assume they are equal. Then ξ includes a uniquely determined map $F_k \longrightarrow X$ and since ξ is a morphism of functors, $F_j \longrightarrow X$ (resp. $F_j' \longrightarrow X$) is obtained by composition with $F_j \longrightarrow F_k$ (resp. $F_j' \longrightarrow F_k$). Therefore the maps $F_i \longrightarrow X$ defined above are the same, namely they are obtained by $F_i \longrightarrow F_k \longrightarrow X$. Done.

PROPOSITION (1.8). Let F, G, $H : I \longrightarrow$ ~~(Sets) be~~ three functors and let $F \longrightarrow G$, $H \longrightarrow G$ be morphisms. Suppose I satisfies (L1, 2, 3). Then

$$\varinjlim (F \times_G H) \xrightarrow{\sim} \varinjlim F \times_{\varinjlim G} \varinjlim H \quad .$$

MORAL (1.9). Axioms (L1, 2, 3) make I as good as an inductive system for limits.

PROPOSITION (1.10). Let I be given, $\mathfrak{F}$ the category of functors $I \longrightarrow (Ab)$. For $F \in \mathfrak{F}$, $\varprojlim F$ is representable and the functor $F \rightsquigarrow \varprojlim F$ is left exact.

Section 2. Presheaves. Let C be a category and denote by $\mathcal{P} = \mathcal{P}_C$ the category of functors $F : C^0 \longrightarrow (Ab)$. We refer to such functors as (abelian) presheaves on C. (The reader may, inserting axioms where necessary, replace (Ab) by an arbitrary category if he feels so inclined.) All of the usual constructions available in (Ab) may be made in $\mathcal{P}$ (by doing them for each $U \in C$) and all functorial properties are preserved. In particular, $\mathcal{P}$ is an abelian category and satisfies axioms AB6, 4^* (cf. Tohoku Proposition 1.6.1). A sequence $F' \longrightarrow F \longrightarrow F'' \in \mathcal{P}$ is exact iff. $F'(U) \longrightarrow F(U) \longrightarrow F''(U)$ is exact for each $U \in C$.

Let C' be another category and $f : C \longrightarrow C'$ a functor. Define a functor $f^p : \mathcal{P}_{C'} = \mathcal{P}' \longrightarrow \mathcal{P}$ by $f^p(F) = F \circ f$. The functor f^p is obviously exact.

Recall that if $M \underset{\beta}{\overset{\alpha}{\rightleftarrows}} N$ are categories and functors then β is left adjoint to α iff. the functors Hom $(Y, \alpha X)$, Hom $(\beta Y, X) : N \times M \longrightarrow$ (Sets) are isomorphic. (If α (resp. β) is given then β (resp. α) is unique up to isomorphism if it exists).

THEOREM (2.1). (Kan) There is a functor $f_p : \mathcal{P} \longrightarrow \mathcal{P}'$ left adjoint to f^p. f_p is right exact.

We sketch the proof for the convenience of the reader:

For $Y \in C'$, let $I_Y = I_Y^f$ be the following category:

(2. 2) $\quad \text{Ob}\, I_Y^f = \{\text{pairs } (X, \phi) \mid X \in \text{ob}\, C \;,\; \phi \in \text{Hom}_{C'}(Y, f(X))\}$

with

$$\text{Hom}((X_1, \phi_1), (X_2, \phi_2)) = \{\xi \in \text{Hom}(X_1, X_2) \mid f(\xi)\phi_1 = \phi_2\}$$

Notice

(a) If $Y \xrightarrow{\epsilon} Z$ in C' we get a functor $\bar{\epsilon} : I_Y \longrightarrow I_Z$ in the obvious way.

(b) If F is a presheaf on C, we get a functor $F_Y : I_Y^0 \longrightarrow (Ab)$ by $F_Y((X, \phi)) = F(X)$. It depends "functorially" on F.

Now set

$$f_p F(Y) = \varinjlim F_Y = \varinjlim_{(X,\phi)\in I_Y} F(X) \quad .$$

$f_p F$ is made into a presheaf on C' (i. e. a functor) by (a). Because of (b), $F \rightsquigarrow f_p F$ is a functor, right exact by Proposition (1. 2). To show f_p is adjoint to f^p we have to define for $F \in \mathcal{P}$, $G \in \mathcal{P}'$ an isomorphism $\text{Hom}(F, f^p G) \xrightarrow{\sim} \text{Hom}(f_p F, G)$ which commutes with maps in $\mathcal{P}$, $\mathcal{P}'$.

Suppose $\xi \in \text{Hom}(F, f^p G)$. Then we have for each $X \in C$ a map $F(X) \longrightarrow G(f(X))$. Hence for $Y \in C'$, $(X, \phi) \in I_Y$ we get a map

$F(X) \longrightarrow G(f(X)) \xrightarrow{G(\emptyset)} G(Y)$, and therefore a map $\varinjlim F_Y = f_p F(Y) \longrightarrow G(Y)$, whence $f_p F \longrightarrow G$. Conversely, if $\eta \in \mathrm{Hom}(f_p F, G)$; i.e., for $Y \in C'$, $f_p F(Y) \longrightarrow G(Y)$ then in particular if $Y = f(X)$ we have $f_p F(f(X)) \longrightarrow G(f(X))$. Now $(X, id_{f(X)}) \in I_{f(X)}$ and so there is a map $F(X) \longrightarrow f_p F(f(X))$. Composing, we find $F(X) \longrightarrow G(f(X))$, whence $F \longrightarrow f^p G$. These maps are obviously inverses of each other and so the theorem follows (of course, some details were omitted).

We want also to recall the following:

COROLLARY (2.3). If f_p is exact then f^p carries injectives into injectives.

For, if $G \in \mathcal{P}'$ is injective, i.e., Hom (x, G) is an exact functor for $x \in \mathcal{P}'$ we want to know Hom $(x, f^p G)$ exact for $x \in \mathcal{P}$. This is the same as Hom $(f_p x, G)$ exact for $x \in \mathcal{P}$. Since f_p is exact and G is injective we are done.

EXAMPLE (2.4). Let $X \in C$ and denote by $\{X\}$ the discrete category $Ob\{X\} = \{X\}$, $Fl\{X\} = \{id_X\}$. Let $i : \{X\} \longrightarrow C$ be the inclusion. A presheaf on $\{X\}$ is just an abelian group, and for $F \in \mathcal{P}_C$, $i^p F = F(X)$. We claim i_p is exact, and to show it we need to show $\varinjlim$ is exact for the categories I^i_Y, $Y \in C$ defined in (2.2). But clearly for $(X, \emptyset)$, $(X, \psi) \in I^i_Y$ Hom $((X, \emptyset), (X, \psi)) = \emptyset$ unless $\emptyset = \psi$ and $= \{id_X\}$ if $\emptyset = \psi$. In other words I^i_Y is the discrete category on the set Hom (Y, X). So if A is an abelian group then $i_p A(Y) = \bigoplus_{\mathrm{Hom}(Y,X)} A$. This is certainly exact.

COROLLARY (2.5). If $F \in \mathcal{P}_C$ is injective then $F(X)$ is an injective abelian group for each $X \in C$.

For, apply (2.3).

NOTATION (2.6). Let $i : \{X\} \longrightarrow C$ be as above. We write $\gamma_X = i_p \mathbb{Z}$, so $\gamma_X \in \mathcal{P}_C$ and we have canonically, for $F \in \mathcal{P}_C$, $\mathrm{Hom}(\gamma_X, F) \approx \mathrm{Hom}(\mathbb{Z}, F(X)) \approx F(X)$, i.e., γ_X represents the (covariant) functor $\mathcal{P} \longrightarrow (\mathrm{Ab})$ given by $F \rightsquigarrow F(X)$. γ_X depends covariantly on $X \in C$. Note that the presheaves γ_X, $X \in C$, are a set of generators for $\mathcal{P}_C$. Hence (cf. Tohoku, Theorem 1.10.1)

COROLLARY (2.7). Every $F \in \mathcal{P}_C$ can be embedded in an injective.

EXAMPLE (2.8). Suppose $C \underset{g}{\overset{f}{\rightleftarrows}} C'$ are given with g left adjoint to f and consider the category I_Y^f, $Y \in C'$. Recall

$$\mathrm{Ob}\, I_Y^f = \{(X, \phi) \mid X \in C,\ \phi : f(X) \longleftarrow Y\} \quad .$$

By adjointness of g to f this is the same as $\{(X, \psi) \mid X \in C, \psi : X \longleftarrow g(Y)\}$. Therefore it is clear that I_Y^f has a final object, namely $(g(Y), \mathrm{id})$, and so if $F \in \mathcal{P}_C$ $f_p F(Y) \approx F(g(Y)) = g^p F(Y)$. Hence $f_p \approx g^p$ and so f^p carries injectives into injectives. This situation arises for instance if one tries to relate C to C/Y ($Y \in C$ fixed), and if $X \times Y$ exists in C, $\forall X$.

Section 3. <u>Czech cohomology.</u> Let C be a category and $\{U_\alpha \longrightarrow V\}_{\alpha \in I}$ a family of maps in C. Suppose the products used below exist in C. We get a lot of maps denoted symbolically by the figure below

$$V \longleftarrow \{U_\alpha\}_{\alpha\in I} \overset{\hat{0}}{\underset{\hat{1}}{\longleftleftarrows}} \{U_\alpha \underset{V}{\times} U_\beta\}_{(\alpha,\beta)\in I^2}$$

$$\overset{\hat{0},\ \hat{1},\ \hat{2}}{\Lleftarrow} \{U_\alpha \underset{V}{\times} U_\beta \underset{V}{\times} U_\gamma\}_{(\alpha,\beta,\gamma)\in I^3} \Lleftarrow \cdots .$$

If F is a presheaf on C we obtain a diagram

$$\prod_\alpha F(U_\alpha) \overset{F(\hat{0})}{\underset{F(\hat{1})}{\rightrightarrows}} \prod_{(\alpha,\beta)} F(U_\alpha \underset{V}{\times} U_\beta)$$

$$\overset{F(\hat{0}),\ F(\hat{1}),\ F(\hat{2})}{\Rrightarrow} \prod_{(\alpha,\beta,\gamma)} F(U_\alpha \underset{V}{\times} U_\beta \underset{V}{\times} U_\gamma) \Rrightarrow \cdots$$

in the canonical way. Define

$$d_n : \prod_{(\alpha_0,\dots,\alpha_n)} F(U_{\alpha_0} \underset{V}{\times} \cdots \underset{V}{\times} U_{\alpha_n}) \longrightarrow \prod_{(\alpha_0,\dots,\alpha_{n+1})} F(U_{\alpha_0} \underset{V}{\times} \cdots \underset{V}{\times} U_{\alpha_{n+1}})$$

by

$$d_n = \sum_{i=0}^{n+1} (-1)^i F(\hat{i}) \quad ,$$

and verify $d_{n+1} \circ d_n = 0$. One obtains a functor $\mathcal{P}_C \longrightarrow$ (cochain complexes) which is obviously exact (since AB4* holds in (Ab)). Taking cohomology, we get an exact sequence of cohomological functors $\mathcal{P}_C \longrightarrow$ (Ab) denoted by $H^q(\{U_\alpha \longrightarrow V\}, \ )$, where

$$H^0(\{U_\alpha \longrightarrow V\}, F) = \ker\left(\prod_\alpha F(U_\alpha) \rightrightarrows \prod_{(\alpha,\beta)} F(U_\alpha \underset{V}{\times} U_\beta)\right) \quad .$$

These functors are called the <u>Czech cohomology</u> of the family $\{U_\alpha \longrightarrow V\}$.

THEOREM (3.1). The functors H^q $(q>0)$ defined above are effacable functors.

COROLLARY (3.2). They are the derived functors of the left exact functor H^0.

<u>Proof of Theorem (3.1):</u> Suppose $F \in \mathcal{P}_C$ is injective and let $\mathcal{Z}_X \in \mathcal{P}_C$ be as in (2.6). We want to show the cochain complex

$$\prod F(U_\alpha) \xrightarrow{d_1} \prod F(U_\alpha \underset{V}{\times} U_\beta) \xrightarrow{d_2} \cdots$$

defined above is exact. This is the same as showing

$$\prod \operatorname{Hom}(\mathcal{Z}_{U_\alpha}, \mathcal{F}) \longrightarrow \prod \operatorname{Hom}(\mathcal{Z}_{U_\alpha \times_V U_\beta}, \mathcal{F}) \longrightarrow \cdots$$

$$=$$

$$\operatorname{Hom}(\oplus \mathcal{Z}_{U_\alpha}, \mathcal{F}) \longrightarrow \operatorname{Hom}(\oplus \mathcal{Z}_{U_\alpha \times_V U_\beta}, \mathcal{F}) \longrightarrow \cdots \text{ exact} \quad .$$

Since $\mathcal{F}$ is injective it suffices to show

$$\oplus \mathcal{Z}_{U_\alpha} \longleftarrow \oplus \mathcal{Z}_{U_\alpha \times_V U_\beta} \longleftarrow \cdots$$

is an exact sequence of presheaves (where the maps are determined by the previous line), i.e., to show that for each $Y \in C$

$$(*) \qquad \oplus \mathcal{Z}_{U_\alpha}(Y) \longleftarrow \bigoplus_{(\alpha,\beta)} \mathcal{Z}_{U_\alpha \times_V U_\beta}(Y) \longleftarrow \cdots$$

is exact. Remembering that $\mathcal{Z}_X(Y) = \bigoplus_{\operatorname{Hom}(Y,X)} \mathbb{Z}$ one sees easily that the sequence (*) is induced by the obvious maps in the diagram

$$\coprod_\alpha \operatorname{Hom}(Y, U_\alpha) \leftleftarrows \coprod_{(\alpha,\beta)} \operatorname{Hom}(Y, U_\alpha \times_V U_\beta) \leftleftarrows \coprod_{(\alpha,\beta,\gamma)} \cdots\cdots$$

$$=$$

$$\coprod_{\phi \in \operatorname{Hom}(Y,V)} \left[\coprod_\alpha \operatorname{Hom}_\phi(Y, U_\alpha) \leftleftarrows \coprod_{(\alpha,\beta)} \operatorname{Hom}_\phi(Y, U_\alpha \times_V U_\beta) \cdots\cdots \right]$$

Set $S(\phi) = \coprod_{\alpha} \mathrm{Hom}_{\phi}(Y, U_\alpha)$.

$$= \coprod_{\phi \in \mathrm{Hom}(Y,V)} \left[S(\phi) \leftleftarrows S(\phi) \times S(\phi) \;\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \cdots \right] .$$

Hence we need to show that a diagram of the form

$$\bigoplus_{S} \mathbb{Z} \leftleftarrows \bigoplus_{S \times S} \mathbb{Z} \;\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \bigoplus_{S \times S \times S} \mathbb{Z} \quad \cdots$$

is exact. But this sequence is homotopically trivial, the homotopy being given by

$$n_{(i_0, \cdots, i_n)} \rightsquigarrow n_{(1, i_0, \cdots, i_n)} \qquad (1 \in S \text{ fixed})$$

for $n_{(\dot{i}_0, \cdots, \dot{i}_n)} = n$ in the $(\dot{i}_0, \cdots, \dot{i}_n)$-th component of $\bigoplus_{S^{n+1}} \mathbb{Z}$.
This completes the proof.

DEFINITION (3.3). Let

$$\{U_\alpha \longrightarrow V\}_{\alpha \in I}, \quad \{U'_\nu \longrightarrow V\}_{\nu \in I'}$$

be families of maps in C. A <u>map</u> $\{U_\alpha \longrightarrow V\} \xrightarrow{f} \{U'_\nu \longrightarrow V\}$ is a map $I \xrightarrow{\epsilon} I'$ and for $\alpha \in I$ a map $U_\alpha \xrightarrow{f_\alpha} U'_{\epsilon(\alpha)}$ such that

$$\begin{array}{ccc} U_\alpha & \longrightarrow & U_{\epsilon(\alpha)} \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

commutes.

One obtains a diagram

$$\begin{array}{ccccccc} V & \longleftarrow & \{U_\alpha\} & \leftleftarrows & \{U_\alpha \underset{V}{\times} U_\beta\} & \Lleftarrow & \cdots\cdots \\ \| & & \Big\downarrow f & & \Big\downarrow f \times f & & \\ V & \longleftarrow & \{U'_\nu\} & \leftleftarrows & \{U'_\nu \underset{V}{\times} U'_\mu\} & \Lleftarrow & \cdots\cdots \end{array}$$

where everything commutes, and hence a map of f^* of complexes

$$\begin{array}{ccccc} \prod F(U_\alpha) & \longrightarrow & \prod F(U_\alpha \underset{V}{\times} U_\beta) & \longrightarrow & \\ \Big\uparrow f_1^* = F(f) & & \Big\uparrow f_2^* = F(f \times f) & & \cdots\cdots \\ \prod F(U'_\nu) & \longrightarrow & \prod F(U'_\nu \underset{V}{\times} U'_\mu) & \longrightarrow & \end{array}$$

which induces a map on Czech cohomology. It is well known that:

PROPOSITION (3.4). If $f, g : \{U_\alpha \longrightarrow V\} \longrightarrow \{U'_\nu \longrightarrow V\}$ are any two maps then f^*, g^* are homotopic, and hence the induced maps on cohomology are equal.

We omit the proof. To get the last fact one has anyhow only to check it for H^0, by Corollary (3. 2), and this is easy.

CHAPTER II. Sheaves

Section 1. Let T be a topology (I. Definition (0.1)), let $\mathcal{P}$ be the category of abelian presheaves on T and $\mathcal{S} \subset \mathcal{P}$ the set of sheaves. We make $\mathcal{S}$ into a full subcategory of $\mathcal{P}$, i.e., we define a morphism of sheaves to be a morphism of presheaves. Denote by $i : \mathcal{S} \hookrightarrow \mathcal{P}$ the inclusion.

THEOREM (1.1). There is a functor $\# : \mathcal{P} \longrightarrow \mathcal{S}$ which is adjoint to i.

If $P \in \mathcal{P}$, $P^{\#} = \#(P)$ is called the associated sheaf to P. Since $\mathcal{S}$ is a subcategory of $\mathcal{P}$ the adjointness property can be stated as follows: There is a functorial homomorphism $P \longrightarrow P^{\#}$ which has the universal mapping property for maps of P into sheaves. The proof is given in the following pages. It can be used without change also for presheaves and sheaves with values in (Sets.)

We begin by defining a functor $+ : \mathcal{P} \longrightarrow \mathcal{P}$: Let U ∈ Cat T be fixed and denote by J_U the category of coverings $\{U_\alpha \longrightarrow U\}$ of U in Cov T, with maps in the sense of I. (3.3). A presheaf $P \in \mathcal{P}$ induces a functor $P_U : J_U^0 \longrightarrow (Ab)$ by

$$P_U(\{U_\alpha \longrightarrow U\}) = \ker \left(\prod P(U_\alpha) \rightrightarrows P(U_\alpha \times_U U_\beta)\right)$$

$$= H^0(\{U_\alpha \longrightarrow U\}, P) \quad .$$

Note that for $V \xrightarrow{\phi} U$ we get also a functor $J_U \xrightarrow{J(\phi)} J_V$ by $\{U_\alpha \longrightarrow U\} \rightsquigarrow \{U_\alpha \underset{U}{\times} V \longrightarrow V\}$ (cf. I. (0.1) (3)), and a morphism of functors $P_U \longrightarrow P_U \circ J(\phi)$ since

$$\begin{array}{ccc}
U_\alpha \underset{U}{\times} V & \leftleftarrows & U_\alpha \underset{U}{\times} U_\beta \underset{U}{\times} V \approx U_\alpha \underset{U}{\times} V \underset{V}{\times} U_\alpha \underset{U}{\times} V \\
\downarrow & & \downarrow \\
U_\alpha & \leftleftarrows & U_\alpha \underset{U}{\times} U_\beta
\end{array}$$

commutes and since ker is a functor. Therefore we get $\varinjlim P_U \longrightarrow \varinjlim P_V$.

Set $P^+(U) = \varinjlim P_U \left(\overset{\text{defn}}{=} \check{H}^0 (T, U ; P)\right)$ and give P^+ the structure of presheaf just defined.

Remark (a): J_U is not a good category for limits, but because of I. (3.4) the homomorphism $P_U(\{U_\alpha \longrightarrow U\}) \longrightarrow P_U(\{V_\nu \longrightarrow U\})$ is uniquely determined if there exist maps $\{V_\nu \longrightarrow U\} \longrightarrow \{U_\alpha \longrightarrow U\}$, and so $P_U : J^0_U \longrightarrow (ab)$ can be factored through $J_U \longrightarrow \bar{J}_U$ where $\bar{J}_U$ is the partially ordered set obtained by writing $\{V_\nu \longrightarrow U\} \geq \{U_\alpha \longrightarrow U\}$ iff. $\exists$ a map $\{V_\nu \longrightarrow U\} \longrightarrow \{U_\alpha \longrightarrow U\}$. $\bar{J}_U$ is an inductive system because (L3) (cf. I. Section 1) holds in J_U. In fact, if $\{U_\alpha \longrightarrow U\}$, $\{V_\nu \longrightarrow U\} \in \operatorname{Cov} T$ then $\{U_\alpha \underset{U}{\times} V_\nu \longrightarrow U\} \in \operatorname{Cov} T$ by axioms (3), (2) of I. (0.1), and we have canonical maps $\{U_\alpha \longrightarrow U\} \longleftarrow \{U_\alpha \underset{U}{\times} V_\nu \longrightarrow U\} \longrightarrow \{V_\nu \longrightarrow U\}$.

It follows, since $\prod$ is exact and ker is left exact, that $+ : \mathcal{P} \longrightarrow \mathcal{P}$ is left exact.

Remark (b): For every $\{U_\alpha \longrightarrow U\} \in J_U$ there is a canonical map $P(U) \longrightarrow P_U(\{U_\alpha \longrightarrow U\})$ given by $P(U) \longrightarrow \prod P(U_\alpha)$. Since (L3) holds in J_U the limit of the constant functor $J_U^0 \longrightarrow (Ab)$ with value $P(U)$ is canonically isomorphic to $P(U)$. We get a functorial homomorphism $P \longrightarrow P^+$ and by definition of sheaf, if $S \in \mathcal{S}$ then $S \longrightarrow S^+$ is bijective. Hence any map of P into a sheaf must factor through P^+. Therefore Theorem(1.1) will be proved if we show

SURPRISE (1.2). P^{++} is a sheaf.

DEFINITION (1.3). Let $P \in \mathcal{P}$. P satisfies (+) iff.

For all $\{U_\alpha \longrightarrow U\} \in \text{Cov } T$, $P(U) \longrightarrow \prod P(U_\alpha)$ is injective.

(1.2) (and hence (1.1)) follows from the following

LEMMA (1.4). (i) P^+ satisfies (+). (ii) If P satisfies (+) then P^+ is a sheaf.

Proof of (i): Let $P \in \mathcal{P}$, $\{U_\alpha \longrightarrow U\} \in \text{Cov } T$ be given. Let $\bar{\xi}_1, \bar{\xi}_2 \in P^+(U)$ and suppose $\bar{\xi}_1, \bar{\xi}_2$ have the same image in $P^+(U_\alpha)$, all α. $\bar{\xi}_1, \bar{\xi}_2$ may be represented by elements

$$\xi_1, \xi_2 \in \ker \left(\prod P(V_\nu) \rightrightarrows \prod P(V_\nu \underset{U}{\times} V_{\nu'})\right)$$

for some $\{V_\nu \longrightarrow U\} \in \mathrm{Cov}\, T$ (we are tacitly using remark (a)). Now the image of $\bar{\xi}_i$ in $P^+(U_\alpha)$ is then represented by the image of ξ_i in

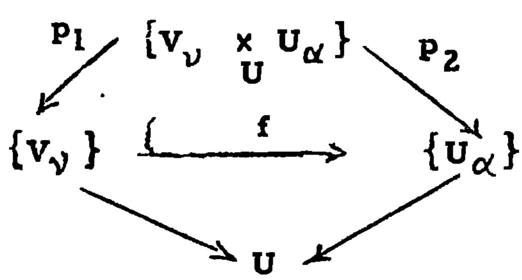
$$\ker \left(\prod_\nu P(U_\alpha \underset{U}{\times} V_\nu) \rightrightarrows \prod_{\nu,\nu'} P(U_\alpha \underset{U}{\times} V_\nu \underset{U}{\times} V_{\nu'})\right) .$$

Since $\bar{\xi}_1 = \bar{\xi}_2$ there is a "finer" covering $\{W_{\alpha\mu} \longrightarrow U_\alpha\}$ such that the images of ξ_1, ξ_2 in $\prod_\mu P(W_{\alpha\mu})$ are equal. But then, letting α vary, the family $\{W_{\alpha\mu} \longrightarrow U\} \in \mathrm{Cov}\, T$ is a refinement of $\{V_\nu \longrightarrow U\}$ and $\xi_1 = \xi_2$ in $\prod_{\alpha,\mu} P(W_{\alpha\mu})$, so $\bar{\xi}_1 = \bar{\xi}_2$.

<u>Proof of (ii):</u> Note first that

(1.5) If P satisfies (+) and if $\{V_\nu \longrightarrow U\} \xrightarrow{f} \{U_\alpha \longrightarrow U\}$ in J_U then the map $\ker \left(\prod P(U_\alpha) \rightrightarrows \prod P(U_\alpha \underset{U}{\times} U_\beta)\right) \longrightarrow \ker \left(\prod P(V_\nu) \rightrightarrows \prod P(V_\nu \underset{U}{\times} V_\mu)\right)$ is injective.

For, consider the diagram



Holding α fixed, $\{V_\nu \underset{U}{\times} U_\alpha \longrightarrow U_\alpha\} \in \mathrm{Cov}\ T$ by axiom (3) of I. (0.1). Hence $\prod_\alpha P(U_\alpha) \hookrightarrow \prod_{\alpha,\nu} P(V_\nu \underset{U}{\times} U_\alpha)$ because of (+). By axiom (2) for T, $\{V_\nu \underset{U}{\times} U_\alpha \longrightarrow U\}_{\alpha,\nu} \in \mathrm{Cov}\ T$. Combining, we find the map

$$\ker\left(\prod P(U_\alpha) \rightrightarrows \prod P(U_\alpha \underset{U}{\times} U_{\alpha'})\right) \longrightarrow \ker\left(\prod P(V_\nu \underset{U}{\times} U_\alpha) \rightrightarrows \quad\right)$$

induced by p_2 is injective. But this map is unique (cf. I. (3.4)) and is therefore the same as the one induced by $p_1 \circ f$. So the one under consideration (induced by f) is also injective.

Now say P satisfies (+), let $\{V_\alpha \longrightarrow U\} \in \mathrm{Cov}\ T$ and

$$\bar{\xi} \in \ker\left(\prod P^+(V_\alpha) \rightrightarrows \prod P^+(V_\alpha \underset{U}{\times} V_\beta)\right) .$$

We have to show $\bar{\xi}$ is image of some element in $P^+(U)$. Choose for each α a family $\{W_{\alpha\nu} \longrightarrow V_\alpha\} \in \mathrm{Cov}\ T$ with $\xi_\alpha \in \ker\left(\prod_\nu P(W_{\alpha\nu}) \rightrightarrows \quad\right)$ representing the α-th component $\bar{\xi}_\alpha$ of $\bar{\xi}$. Consider the diagram (in which all squares are cartesian)

$$
\begin{array}{ccccc}
\{W_{\alpha\nu}\} & \longleftarrow & \{W_{\alpha\nu} \underset{U}{\times} V_\beta\} & \longleftarrow & \{W_{\alpha\nu} \times W_{\beta\mu}\} \\
\downarrow & & \downarrow & & \downarrow \\
\{V_\alpha\} & \longleftarrow & \{V_\alpha \underset{U}{\times} V_\beta\} & \longleftarrow & \{V_\alpha \underset{U}{\times} W_{\beta\mu}\} \\
\downarrow & & \downarrow & & \downarrow \\
U & \longleftarrow & \{V_\beta\} & \longleftarrow & \{W_{\beta\mu}\}
\end{array}
$$

We find ξ_α induces by base extension an element

$$\xi^1_{\alpha\beta} \in \ker\left(\prod_\nu P(W_{\alpha\nu} \underset{U}{\times} V_\beta) \rightrightarrows \prod_{\nu,\nu'} P\left(W_{\alpha\nu} \times V_\beta \underset{V_\alpha \underset{U}{\times} V_\beta}{} W_{\alpha\nu'} \underset{U}{\times} V_\beta\right)\right)$$

and ξ_β induces an element

$$\xi^2_{\alpha\beta} \in \ker\left(\prod_\mu P(V_\alpha \underset{U}{\times} W_{\beta\mu}) \rightrightarrows \prod_{\mu,\mu'} P\left(V_\alpha \underset{U}{\times} W_{\beta\mu} \underset{V_\alpha \underset{U}{\times} V_\beta}{\times} V_\alpha \underset{U}{\times} W_{\beta\mu'}\right)\right).$$

By assumption on $\bar{\xi}$, $\xi^1_{\alpha\beta}$ and $\xi^2_{\alpha\beta}$ represent the same element of $P^+(V_\alpha \underset{U}{\times} V_\beta)$. Hence "$\xi^1_{\alpha\beta} = \xi^2_{\alpha\beta}$" in some covering of $V_\alpha \times V_\beta$

which is a common refinement of $\{W_{\alpha\nu} \underset{U}{\times} V_\beta \longrightarrow V_\alpha \underset{U}{\times} V_\beta\}_\nu$ and $\{V_\alpha \underset{U}{\times} W_{\beta\mu} \longrightarrow V_\alpha \underset{U}{\times} V_\beta\}_\mu$. By (1.5) this must be so in <u>any</u> common refinement, and hence " $\xi^1_{\alpha\beta} = \xi^2_{\alpha\beta}$" in $\prod_{\nu,\mu} P(W_{\alpha\nu} \underset{U}{\times} W_{\beta\mu})$. This shows

$$\xi \in \ker\Big(\prod_{\alpha,\nu} P(W_{\alpha\nu}) \rightrightarrows \prod_{\alpha,\nu,\beta,\mu} P(W_{\alpha\nu} \underset{U}{\times} W_{\beta\mu})\Big), \text{ whence } \bar{\xi} \in P^+(U)$$

Having theorem (1.1) one can copy large parts of Godement's book. In particular,

THEOREM (1.6). (i) $\mathcal{S}$ is an abelian category satisfying AB5, AB3* and has generators. (ii) $i : \mathcal{S} \hookrightarrow \mathcal{P}$ is left exact and $\# : \mathcal{P} \longrightarrow \mathcal{S}$ is exact.

<u>Proof of (i):</u> Let $F \longrightarrow G \in \mathcal{S}$, and set K = "Presheaf cokernel". Then if $\{U_\alpha \longrightarrow U\} \in \mathrm{Cov}\, T$ we have

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & K(U) & \longrightarrow & \prod K(U_\alpha) & \rightrightarrows & \prod K(U_\alpha \underset{U}{\times} U_\beta) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F(U) & \longrightarrow & \prod F(U_\alpha) & \rightrightarrows & \prod F(U_\alpha \underset{U}{\times} U_\beta) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G(U) & \longrightarrow & \prod G(U_\alpha) & \rightrightarrows & \prod G(U_\alpha \underset{U}{\times} U_\beta)
\end{array}$$

.

Since ker is a left exact functor in (Ab) it follows that

$$K(U) \xrightarrow{\sim} \ker\left(\prod K(U_\alpha) \rightrightarrows \prod K(U_\alpha \times_U U_\beta)\right) ,$$

i.e., that K is a sheaf. Obviously, therefore, K is a kernel in $\mathcal{S}$. (i.e. $0 \longrightarrow \mathrm{Hom}(X,K) \longrightarrow \mathrm{Hom}(X,F) \longrightarrow \mathrm{Hom}(X,G)$ is exact, $X \in \mathcal{S}$).

Let C = "presheaf coker ($F \longrightarrow G$)". Then for $X \in \mathcal{S}$ we have (by the universal mapping property of $P \longrightarrow P^{\#}$)

$$\begin{array}{ccccccc}
\mathrm{Hom}(F,X) & \longleftarrow & \mathrm{Hom}(G,X) & \longleftarrow & \mathrm{Hom}(C,X) & \longleftarrow & 0 \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\
\mathrm{Hom}(F^{\#},X) & \longleftarrow & \mathrm{Hom}(G^{\#},X) & \longleftarrow & \mathrm{Hom}(C^{\#},X) & \longleftarrow & 0
\end{array} .$$

Here the top line is exact by definition of C, and since F, G are sheaves, $F \approx F^{\#}$ and $G \approx G^{\#}$. Therefore $C^{\#}$ (with the induced map $G \longrightarrow C^{\#}$) is a cokernel in $\mathcal{S}$.

Finally, let I = "presheaf coker ($K \longrightarrow F$)" = "presheaf coimage ($F \longrightarrow G$)". Then $I^{\#}$ = "sheaf coimage ($F \longrightarrow G$)" with the obvious map. We need to show $I^{\#}$ is the image = $\ker(G \longrightarrow C^{\#})$. Now certainly $0 \longrightarrow I \longrightarrow G \longrightarrow C$ is exact in $\mathcal{P}$ since $\mathcal{P}$ is an abelian category. Therefore, since $i \circ \# : \mathcal{P} \longrightarrow \mathcal{P}$ is left exact because + is (cf. remark (a)),

$0 \longrightarrow I^{\#} \longrightarrow G^{\#} \longrightarrow C^{\#}$ is exact in $\mathcal{P}$. Recalling the discussion of ker above and that $G \xrightarrow{\sim} G^{\#}$ we are done.

<u>AB3* (existence of product):</u> Let F_i, $i \in I$ be sheaves. It is immediate that the presheaf product of the F_i s is a sheaf, and hence a fortiori a product in the category of sheaves.

<u>AB5 (existence of sums, etc.):</u> Let F_i, $i \in I$ be sheaves, and F = "presheaf sum" of the F_i s. By the universal mapping property of $\#$, it is clear that $F^{\#}$ is a sum in the category of sheaves.

Let C, B be sheaves, and $A_i \subset C^{(i \in I)}$ an increasing filtering family of subsheaves. Suppose given morphisms $A_i \longrightarrow B$ compatible with the inclusions of A_i in A_j $(i \leq j)$. To complete the proof of AB5 we have to find a map $A \longrightarrow B$ inducing the maps $A_i \longrightarrow B$, where $A = \sup A_i \subset C$. Set $\overline{A}(U) = \bigcup_i A_i(U)$, whence $\overline{A} \subset C$ is a presheaf. Since C is a sheaf, the inclusion of $\overline{A}$ can be factored through $\overline{A} \longrightarrow \overline{A}^{\#} \longrightarrow C$, and since $\#$ is left exact, $\overline{A}^{\#} \hookrightarrow C$. Obviously $\overline{A}^{\#}$ (with this injection) is canonically isomorphic to A. Since the maps $A_i \longrightarrow B$ define a map $\overline{A} \longrightarrow B$ and since B is a sheaf, we find the desired map $\overline{A}^{\#} \longrightarrow B$.

<u>Generators:</u> Cleary $\mathcal{J}_U^{\#}$ is a set of generators for $\mathcal{S}$ (cf. I. (2. 6)).

NOTATION (1. 7). We write $\mathcal{J}_U^{\#} = \mathbb{Z}_U$. $\mathbb{Z}_U$ represents the (covariant) functor $F \rightsquigarrow F(U)$ on $\mathcal{S}$.

<u>Proof of (ii):</u> $i : \mathcal{S} \longrightarrow \mathcal{P}$ is left exact because the presheaf kernel of a map of sheaves is already a sheaf. $\# : \mathcal{P} \longrightarrow \mathcal{S}$ is left exact because $i \circ \#$ is (as was noted above) and because i is left exact and fully faithful. To show $\#$ right exact, let $P' \longrightarrow P \longrightarrow P'' \longrightarrow 0 \in \mathcal{P}$ be exact, and $X \in \mathcal{S}$. Then

$$\begin{array}{ccccccc}
\mathrm{Hom}(P', X) & \longleftarrow & \mathrm{Hom}(P, X) & \longleftarrow & \mathrm{Hom}(P'', X) & \longleftarrow & 0 \\
\wr\wr & & \wr\wr & & \wr\wr & & \\
\mathrm{Hom}(P'^{\#}, X) & \longleftarrow & \mathrm{Hom}(P^{\#}, X) & \longleftarrow & \mathrm{Hom}(P''^{\#}, X) & \longleftarrow & 0
\end{array}$$

commutes, and the top line is exact, hence the bottom is, and $P'^{\#} \longrightarrow P^{\#} \longrightarrow P''^{\#} \longrightarrow 0$ is exact.

MISCELLANY (1. 8):

(i) The category $\mathcal{S}$ has enough injectives.

(ii) An injective in $\mathcal{S}$ is injective as presheaf.

(iii) For $U \in \mathrm{Cat}\ T$, the functor $\Gamma_U : \mathcal{S} \longrightarrow (\mathrm{Ab})$ given by $\Gamma_U(F) = F(U)$ is left exact.

(iv) Let $0 \longrightarrow F \longrightarrow G \in \mathcal{S}$ be exact, and C = "presheaf cokernel $(F \longrightarrow G)$". Then C satisfies (+).

Proof:

(i) cf. Tohoku, Theorem 1.10.1.

(ii) Because # is exact (cf. I. (2.3)).

(iii) Γ_U is the composition of the left exact functor i and the exact section functor on $\mathcal{P}$.

(iv) We have a diagram, for $\{U_i \longrightarrow U\} \in \mathrm{Cov}\ T$:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(U) & \longrightarrow & \prod F(U_i) & \rightrightarrows & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G(U) & \longrightarrow & \prod G(U_i) & \rightrightarrows & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker & \longrightarrow & \prod C(U_i) & \rightrightarrows & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

Hence $C(U) = G(U)/F(U) \hookrightarrow \ker \left(\prod C(U_i) \rightrightarrows \qquad \right)$.

HOMEWORK (1.9). Discuss $\underline{\mathrm{Hom}}$ (F, G).

Section 2. Cohomology. Let $f : \mathcal{S} \longrightarrow \mathcal{Q}$ be any left exact functor, where $\mathcal{S}$ is as above and $\mathcal{Q}$ is an abelian category. Because of (1.8)(i) one can define the right derived functors $R^q f$ of f (cf. Tohoku, Section 2.3).

NOTATION (2.1). Let $U \in \mathrm{Cat}\ T$. The derived functors of $\Gamma_U : \mathcal{S} \longrightarrow (\mathrm{Ab})$ (cf. (1.8) (iii)) are denoted by $R^q \Gamma_U = H^q(T, U; \ )$. So in particular, for $F \in \mathcal{S}$, $H^0(T, U; F) = F(U)$.

NOTATION (2.2). Fixing $F \in \mathcal{S}$, we get a functor $\Gamma(F) : \mathrm{Cat}\ T^0 \longrightarrow (\mathrm{Ab})$ by $U \rightsquigarrow F(U) = \Gamma_U(F)$. Denote by $\underleftarrow{\Gamma} : \mathcal{S} \longrightarrow (\mathrm{ab})$ the functor $F \rightsquigarrow \varprojlim \Gamma(F)$ $= \varprojlim_{U \in \mathrm{Cat}\ T^0} F(U)$. Because of I, Proposition (1.10), it is clear that $\underleftarrow{\Gamma}$ is a left exact functor. The right derived functors are denoted by $R^q \underleftarrow{\Gamma} = H^q(T, \ )$.

Clearly, if Cat T has a final object X (initial in Cat T^0), then $H^q(T, F) \approx H^q(T, X; F)$, $F \in \mathcal{S}$.

NOTATION (2.3). The right derived functors of $i : \mathcal{S} \hookrightarrow \mathcal{P}$ are denoted by $R^q i = \mathcal{H}^q(\ )$.

So for $F \in \mathcal{S}$ the $\mathcal{H}^q(F)$ are presheaves, and $\mathcal{H}^0(F) = F$ viewed as a presheaf. In fact, we have canonically

$$(2.4) \qquad [\mathcal{H}^q(F)](U) \approx H^q(T, U; F) \quad .$$

To show this we remark that $H^q(T, \; ; F)$ is a functor of U (i.e., a presheaf) since F is . To show $H^q(T, \; ; F)$ are the derived functors of i we have only to check

(a) They agree if $q = 0$.

(b) They vanish on injectives for $q > 0$.

(c) They form an exact cohomological functor.

All of these assertions are trivial.

PROPOSITION (2.5): $(\mathcal{H}^q(F))^+ = 0$ for $F \in \mathcal{S}$, $q > 0$.

Proof: It follows from Lemma (1.4) that for $P \in \mathcal{P}$, $P^+ \hookrightarrow P^{\#}$. Hence we need only show $(\mathcal{H}^q(F))^{\#} = 0$. Now $\# \circ i \approx id_{\mathcal{S}}$. The functor $\#$ is exact (by (1.6)) and so its derived functors vanish. We get a spectral sequence for the composed functor (cf. Tohoku, Theorem (2.4.1)) from which one can read off the desired result.

Section 3. The spectral sequence for Czech cohomology. Let $\{U_\alpha \longrightarrow U\} \in \mathrm{Cov}\ T$, and $F \in \mathcal{S}$. Then F(U) is functorially isomorphic to $H^0(\{U_\alpha \longrightarrow U\}, F)$ (cf. I. Section 3), by definition of sheaf. In other words, the functor Γ_U defined above is isomorphic to the composition of the functors i, $H^0(\{U_\alpha \longrightarrow U\}, \;)$. Since i carries injectives into injectives ((1.8) (ii)) we obtain (cf. Tohoku, Theorem 2.4.1).

SPECTRAL SEQUENCE (3.1).

$$E_2^{p,q} = H^p(\{U_\alpha \longrightarrow U\}, \mathcal{H}^q(F)) \Longrightarrow H^*(T, U; F) \quad (F \in \mathcal{S})$$

More generally, let $U \in \text{Cat}\, T$ be fixed, and $J \subset J_U$ be any subcategory satisfying (L3), where J_U is as in Section 1. Denote by $H^0(J; \ ) : \mathcal{P} \longrightarrow (ab)$ the functor

$$H^0(J; P) = \varinjlim_{\{U_\alpha \longrightarrow U\} \in J} H^0(\{U_\alpha \longrightarrow U\}; P) \quad .$$

By reasoning similar to that of remark (a) of Section 1, one sees that $H^0(J; \ )$ is a left exact functor and that its right derived functors $H^q(J; \ )$ are obtained as

$$(3.2) \qquad H^q(J; P) = \varinjlim_{\{U_\alpha \longrightarrow U\} \in J} H^q(\{U_\alpha \longrightarrow U\}; P).$$

(One uses Corollary (3.2) and Proposition (3.4) of Chapter I.) Moreover Γ_U is isomorphic to the composition of the functors i and $H^0(J; \ )$. Hence we obtain for $F \in \mathcal{S}$.

SPECTRAL SEQUENCE (3.3).

$$E_2^{p,q} = H^p(J; \mathcal{H}^q(F)) \Longrightarrow H^*(T, U; F). \qquad (F \in \mathcal{S})$$

In the special case $J = J_U$ is the category of all coverings of U, we write

NOTATION (3.4)

$$\check{H}^0(T, U; P) = H^0(J_U; P) = P^+(U) .$$

$$\check{H}^q(T, U; P) = H^q(J_U; P) \qquad (P \in \mathcal{P}') .$$

$\check{H}^q(T, U; P)$ is called the Czech cohomology of P with respect to U. Note that although it is a functor on $\mathcal{P}'$ (which depends only on Cat T), the Czech cohomology depends also on Cov T. (3.3) now reads, for $F \in \mathcal{S}$.

SPECRTAL SEQUENCE (3.5).

$$E_2^{p,q} = \check{H}^p(T, U; \mathcal{H}^q(F)) \Longrightarrow H^*(T, U; F) \qquad (F \in \mathcal{S}).$$

Since $\check{H}^0(T, U; P) = P^+(U)$, it follows from Proposition (2.5) that $E_2^{0,q} = 0$, $q > 0$. Therefore one reads from (3.5)

COROLLARY (3.6).

$$\check{H}^1(T, U; F) \approx H^1(T, U; F)$$

$$\check{H}^2(T, U; F) \hookrightarrow H^2(T, U; F) \qquad (F \in \mathcal{S}).$$

Section 4. The Category (Top). Leray Spectral sequences.

Recall

LEMMA (4.1) (Tohoku). Let $f : \mathcal{A} \longrightarrow \mathcal{A}'$ be a left exact functor with $\mathcal{A}$, $\mathcal{A}'$ abelian categories and suppose $\mathcal{A}$ has enough injectives. If $\mathcal{M} \subset \mathrm{Ob}\,\mathcal{A}$ satisfies

(a) For all $F \in \mathcal{A}$, $\exists$ $0 \longrightarrow F \longrightarrow M$ exact in $\mathcal{A}$ with $M \in \mathcal{M}$.

(b) $F \oplus G \in \mathcal{M} \Longrightarrow F \in \mathcal{M}$.

(c) If $0 \longrightarrow M' \longrightarrow M \longrightarrow F \longrightarrow 0$ exact in $\mathcal{A}$, M', $M \in \mathcal{M}$ then $F \in \mathcal{M}$ and $f\,(0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0)$ is exact,

then all injectives are in $\mathcal{M}$ and $R^q f(M) = 0$, all $M \in \mathcal{M}$, $q > 0$. (Hence resolutions from $\mathcal{M}$ may be used to calculate the functors $R^q f$.)

DEFINITION (4.2). Let T be a topology and F a sheaf on T. F is <u>flask</u> iff. for every $\{U_\alpha \longrightarrow U\} \in \mathrm{Cov}\,T$, $H^q(\{U_\alpha \longrightarrow U\}, F) = 0$, $q > 0$ (cf. I, Section 3).

PROPOSITION (4.3). Let $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ be an exact sequence of sheaves. Then

(i) If F' is flask, the sequence is exact as a sequence of presheaves.

(ii) F', F flask $\Longrightarrow$ F'' flask.

(iii) $F \oplus G$ flask $\Longrightarrow$ F flask.

(iv) Injectives are flask.

<u>Proof:</u> (i): Say F' flask. To show the sequence exact as a sequence of presheaves it suffices to show the presheaf coker $(F' \longrightarrow F) = C$ is a sheaf, as then $C \approx C^{\#} \approx F''$. We have to show for $\{U_\alpha \longrightarrow U\} \in \mathrm{Cov}\, T$ that $C(U) \longrightarrow \ker \left(\prod C(U_\alpha) \rightrightarrows \prod C(U_\alpha \times_U U_\beta)\right) = H^0(\{U_\alpha \longrightarrow U\}; C)$ is bijective.

This is obvious from the diagram

$$\begin{array}{ccccccccc}
0 \rightarrow H^0(\{U_\alpha \rightarrow U\}; F') & \rightarrow & H^0(\{U_\alpha \rightarrow U\}, F) & \rightarrow & H^0(\{U_\alpha \rightarrow U\}, C) & \rightarrow & H^1(\{U_\alpha \rightarrow U\}, F') \\
\uparrow\wr & & \uparrow\wr & & \uparrow\wr & & \| \\
0 \rightarrow F'(U) & \longrightarrow & F(U) & \longrightarrow & C(U) & \longrightarrow & 0
\end{array}$$

since $H^1(\{U_\alpha \longrightarrow U\}, F') = 0$ by assumption.

(ii): Knowing (i), this is clear from the exact cohomology sequences for $H^q(\{U_\alpha \longrightarrow U\}, \ \ )$.

(iii): The cohomology commutes with finite direct sums.

(iv): Trivial.

COROLLARY (4.4). Flask resolutions can be used to calculate the derived functors $H^q(T, U; \ )$ and $\mathcal{H}^q(\ )$ (cf. (2.1), (2.3)).

In fact, applying (4.3) to (4.1) we find (iv) $\Longrightarrow$ (a), (iii) $\Longrightarrow$ (b) and (i), (ii) $\Longrightarrow$ (c) if $f(F) = F(U)$ or $i(F)$. It is <u>not</u> true in general that flask resolutions can be used to calculate $H^q(T, \ )$ (cf. (2.2)).

DEFINITION (4.5). We define the category (Top) of topologies as follows: Ob (Top) = set of topologies, and, for $T, T' \in$ Ob (Top), Hom (T, T') = set of functors f: Cat $T \longrightarrow$ Cat T' satisfying:

If $\{U_\alpha \xrightarrow{\varphi_\alpha} U\} \in$ Cov T and $V \longrightarrow U \in$ Cat T then

(i) $\{(U_\alpha) \xrightarrow{f(\varphi_\alpha)} f(U)\} \in$ Cov T'.

(ii) $f(U_\alpha \times_U V) \xrightarrow{\sim} f(U_\alpha) \times_{f(U)} f(V)$ for all α (with the canonical map

A functor $f \in$ Hom (T, T') is called a <u>morphism</u> of topologies.

EXAMPLE (4.6). Let T, T' be the topologies on topological spaces X, X' respectively. A continuous map $\pi : X' \longrightarrow X$ induces, as is well known, a morphism $f : T \longrightarrow T'$. If $X' \subset X$ is an open subset, then the inclusion $T' \subset T$ is a morphism, adjoint in fact to the restriction $T \longrightarrow T'$ above.

Let $T, T' \in$ Ob(Top). Denote by $\mathcal{S}_T, \mathcal{S}_{T'}, \mathcal{P}_T, \mathcal{P}_{T'}$ the categories of sheaves and presheaves on T, T' and by $i, i', \#, \#'$ the functors of Section 1. Let $f : T \longrightarrow T'$ be a morphism. It is immediate from the definition above that if $F' \in \mathcal{S}_{T'}$ then $f^p F' \in \mathcal{S}_T$ (where f^p is as defined in I. Section 2). We get a functor

$$f^s : \mathcal{S}_{T'} \longrightarrow \mathcal{S}_T \quad ; \quad f^s = f^p \circ i' \approx \# \circ f^p \circ i'$$

which is clearly left exact. So we have for $F' \in \mathcal{S}_{T'}$ and $U \in \text{Cat } T$, $f^s F'(U) = F'(f(U))$. One verifies readily that f^s has as left adjoint

$$f_s : \mathcal{S}_T \longrightarrow \mathcal{S}_{T'} \quad ; \quad f_s = \#' \circ f_p \circ i \quad .$$

Note however: It is not clear that f_s is exact, even if f_p is.

The right derived functors $R^q f^s$ of f^s can be described as follows: $\#$ and f^p are exact, hence $\# \circ f^p$ is exact, and so its derived functors vanish. Therefore, since $f^s \approx \# \circ f^p \circ i'$ we have, for $F' \in \mathcal{S}_{T'}$ and $\mathcal{H}^q(F') = R^q i' F'$,

$$R^q f^s(F') \approx \left[f^p (\mathcal{H}^q (F'))\right]^{\#} \qquad \text{(here p is not an intege}$$

Or, writing $\mathcal{R}^q F' = f^p(\mathcal{H}^q(F'))$ and referring to (2.4) we get

COROLLARY (4.7): $R^q f^s(F') \approx [\mathcal{R}^q F']^{\#}$ where for $U \in \text{Cat } T$ $[\mathcal{R}^q F'](U) = H^q(T', f(U) ; F')$.

LEMMA (4.8). Let $f \in \mathrm{Hom}(T, T')$ and $F' \in \mathcal{S}_{T'}$. If F' is flask, so is $f^s F'$.

Proof: Let $\{U_\alpha \longrightarrow U\} \in \mathrm{Cov}\, T$. Applying (4.5) (ii) repeatedly, we find $f(U_{\alpha_0} \underset{U}{\times} \cdots \underset{U}{\times} U_{\alpha_n}) \xrightarrow{\sim} f(U_{\alpha_0}) \underset{f(U)}{\times} \cdots \underset{f(U)}{\times} f(U_{\alpha_n})$ (where the map is canonical). Therefore

$$\prod_{(\alpha_0, \cdots, \alpha_n)} f^s F'(U_{\alpha_0} \underset{U}{\times} \cdots \underset{U}{\times} U_{\alpha_n}) = \prod_{(\alpha_0, \cdots, \alpha_n)} F'\Big(f(U_{\alpha_0} \underset{U}{\times} \cdots \underset{U}{\times} U_{\alpha_n})\Big)$$

$$\xleftarrow{\sim} \prod_{(\alpha_0, \cdots, \alpha_n)} F'\Big(f(U_{\alpha_0}) \underset{f(U)}{\times} \cdots \underset{f(U)}{\times} f(U_{\alpha_n})\Big)$$

This isomorphism commutes with the projection maps and so the lemma follows from (4.5) (i) and the definition of flask.

COROLLARY (4.9). Let $T'' \xrightarrow{g} T \xrightarrow{f} T'$ be morphisms of topologies, and $F' \in \mathcal{S}_{T'}$ a flask sheaf. Then $f^s F'$ is $R^q g^s$-acyclic.

This is immediate from the lemma and from (4.7) applied to g.

Since $(fg)^s = g^s f^s$ we obtain:

SPECTRAL SEQUENCE (4.10). $T'' \xrightarrow{g} T \xrightarrow{f} T'$ morphisms of topologies, $F' \in \mathcal{S}_{T'}$:

$$E_2^{p,q} = R^p g^s(R^q f^s(F')) \Longrightarrow R^*(fg)^s(F').$$

In the special case T'' is the discrete category $\{X\}$ (there is only one topology), $X \in \text{Cat } T$, and g is the inclusion map we have $R^p q^s F = H^p(T, X; F)$ $(F \in \mathcal{S}_T)$ and $R^p (fg)^s F' = H^p(T', f(X); F')$ $(F' \in \mathcal{S}_{T'})$. Hence

SPECTRAL SEQUENCE (4.11). $T \xrightarrow{f} T'$ a morphism of topologies, $X \in \text{Cat } T$, $Y = f(X)$, and $F' \in \mathcal{S}_{T'}$.

$$E_2^{p,q} = H^p(T, X; R^q f^s (F')) \Longrightarrow H^*(T', Y; F') \quad .$$

Note: In case f is obtained from a map $\pi: X' \longrightarrow X$ of topological spaces as in (4.6) (so that $Y = X'$) this is the usual Leray spectral sequence. However, one needs much stronger conditions on f than those of (4.5) to get a spectral sequence relating $H^q(T, \ )$ and $H^q(T', \ )$ (cf. (2.2)) in general. For one thing, $\Gamma \circ f^s$ need not be isomorphic to the functor $\Gamma' = H^0(T', \ )$.

DEFINITION (4.12): Let T be a topology, $Y \in \text{Cat } T$. Define a topology T/Y by $\text{Cat } T/Y = (\text{Cat } T)/Y$ and $\text{Cov } T/Y$ = set of families of maps $\{U_\alpha \longrightarrow U\}$ <u>over Y</u> such that $\{U_\alpha \longrightarrow U\} \in \text{Cov } T$.

Let $\epsilon: \text{Cat } T/Y \longrightarrow \text{Cat } T$ be the functor "ignore Y". ϵ is a morphism of topologies and it is trivial that ϵ^s is exact. Applying (4.11).

and taking into account the fact that (Y, id_Y) is final in Cat T/Y we find (elucidating the functors (2.1))

COROLLARY (4.13). Let T be a topology, Y $\in$ Cat T, and $\epsilon : T/Y \longrightarrow T$ the canonical morphism. Let $F \in \mathcal{S}_T$ and $U \xrightarrow{\phi} Y \in$ Cat T/Y. Then

$$H^q(T, U; F) \approx H^q(T/Y, (U, \phi); \epsilon^s F) .$$

In particular,

$$H^q(T, Y; F) \approx H^q(T/Y, (Y, id); \epsilon^s F) \approx H^q(T/Y; \epsilon^s F)$$

THEOREM (4.14). Let $f : T \longrightarrow T'$ be a morphism and suppose

(i) Cat T and Cat T' have final objects and finite fibered products.

(ii) f preserves final objects and finite fibered products (i.e. f is left exact).

Then f_s is exact.

Proof: We first show f_p is exact. It suffices to show that the categories I_V^f, V $\in$ Cat T' (cf. I. (2.2)) satisfy axioms $(L1, 2, 3)^0$ dual to(L1, 2, 3) of I. Section I.

$(L1)^0$: If

$$\begin{array}{ccccc} (U_2, \phi_2) & & & & (U_3, \phi_3) \\ & \searrow & & \swarrow & \\ & & (U_1, \phi_1) & & \end{array}$$

is given in I_V^f, i.e., a diagram

$$\begin{array}{ccccc} U_2 & & & & U_3 \\ & \searrow & & \swarrow & \\ & & U_1 & & \end{array}$$

in Cat T such that

$$\begin{array}{ccccc} & & V & & \\ & {}^{\phi_2}\swarrow & \downarrow{\scriptstyle \phi_1} & \searrow^{\phi_3} & \\ f(U_2) & & & & f(U_3) \\ & \searrow & & \swarrow & \\ & & f(U_1) & & \end{array}$$

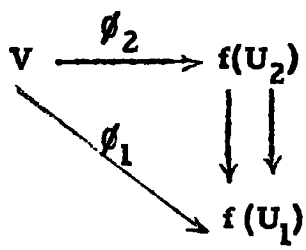
commutes, take the induced map

$$V \longrightarrow f(U_2 \underset{U_1}{\times} U_3) \approx f(U_2) \underset{f(U_1)}{\times} f(U_3) \quad .$$

$(L2)^0$: Given $(U_2, \phi_2) \rightrightarrows (U_1, \phi_1)$, i.e., a diagram

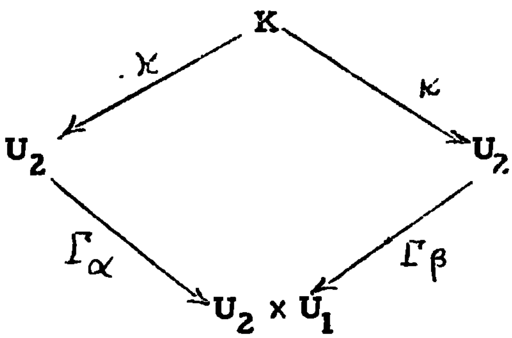
$$U_2 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} U_1$$

in Cat T such that the two triangles in the diagram



commute,

denote by $K \xrightarrow{\varkappa} U_2$ the kernel of the pair (α, β) of maps, i.e., K is the object making the diagram below cartesian



(where the product $U_2 \times U_1$ is over the final object).

By assumption (ii), $f(K) \longrightarrow f(U_2) \rightrightarrows f(U_1)$ is again exact, so there exists a canonical map ψ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ \psi\ } & f(K) \\ & \searrow & \downarrow \\ & & f(U_2) \end{array} \qquad \text{commutes .}$$

Hence we get $(K, \psi) \longrightarrow (U_2, \phi_2)$ which is as required.

$(L3)^0$ Given $(U_1, \phi_1), (U_2, \phi_2)$ take $(U_1 \times U_2, \phi_1 \times \phi_2)$.

Now we claim that, whenever f is such that the categories I^f_V $(V \in \text{Cat } T')$ satisfy $(L1, 2, 3)^0$, f_s is exact: Let $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ be exact in $\mathcal{S}_T$, and denote by C the presheaf cokernel $(F \longrightarrow F'')$. Since f_p is exact, the sequence $\#' \cdot f_p[0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow C \longrightarrow 0]$ is exact in $\mathcal{S}_{T'}$ and so we have to show $(f_p C)^\# = 0$. Let $V \in \text{Cat } T'$ and

$$\bar{\xi} \in f_p\, C(V) = \varinjlim_{(U, \phi) \in I^f_V} C(U) \quad ,$$

and choose $\xi \in C(U)$, $(U, \phi) \in I^f_V$ representing $\bar{\xi}$ (we are tactily using the discussion of I. Section 1, notably (1.3)). Since $C^\#$, and hence C^+, is the zero sheaf, we can find $\{U_\alpha \longrightarrow U\} \in \text{Cov } T$ such that $\xi \rightsquigarrow 0$ in $\prod C(U_\alpha)$. Consider the diagram

$$\begin{array}{ccc} V & \longleftarrow & V_\alpha = V \times_{f(U)} f(U_\alpha) \\ \phi \downarrow & \overset{\phi_\alpha}{\swarrow} & \downarrow \psi_\alpha \\ f(U) & \longleftarrow & f(U_\alpha) \end{array}$$

where ϕ_α is the composition.

In the functor $I^f_V \longrightarrow I^f_{V_\alpha}$ induced the map $V \longleftarrow V_\alpha$ we have $(U, \phi) \rightsquigarrow (U, \phi_\alpha)$. Also, in $I^f_{V_\alpha}$ there is a map $(U, \phi_\alpha) \longleftarrow (U_\alpha, \psi_\alpha)$ given by $U \longleftarrow U_\alpha$. Hence since $\xi \rightsquigarrow 0$ in $C(U_\alpha)$ it follows that $\bar{\xi} \rightsquigarrow 0$ in $f_p\, C(V_\alpha)$. But $\{V_\alpha \longrightarrow V\}$ is obtained by base extension from a covering, hence is in Cov T'. Since $\bar{\xi} \rightsquigarrow 0$ in $\prod f_p\, C(V_\alpha)$ it follows $\bar{\xi} \rightsquigarrow 0$ in $C^+(V)$, whence $C^+ = 0$.

Section 5. Inductive limits for noetherian topologies. Let T be a topology and $\mathcal{P}$, $\mathcal{S}$ the categories of presheaves and sheaves on T. If I i a category we write $\mathcal{P}^I = \mathrm{Hom}_{Cat}(I, \mathcal{P})$, $\mathcal{S}^I = \mathrm{Hom}_{Cat}(I, \mathcal{S})$. For $P \in \mathcal{P}^I$, $\varinjlim P$ is representable. In fact we need only take $[\varinjlim P](U) = \varinjlim [P(U)]$. $\varinjlim : \mathcal{P}^I \longrightarrow \mathcal{P}$ is exact if I satisfies (L1, 2) (cf I. Section 1).

Let $F \in \mathcal{S}^I$ and let p. $\varinjlim F$ denote the limit in the category of presheaves. Clearly, $(p. \varinjlim F)^\# = \varinjlim F$ represents the functor $\varinjlim F$. We claim if I satisfies (L1, 2) then $\varinjlim : \mathcal{S}^I \longrightarrow \mathcal{S}$ is exact. In fact, writing $I = \coprod_\nu I_\nu$ as direct sum of its connected components, and recalling that AB5 holds in $\mathcal{S}$, one reduces to the case I satisfies (L1, 2, 3 Moreover, $\varinjlim$ is certainly left exact since we have $\varinjlim = \# \circ \underline{p. \lim} \circ i^I$ $(i^I : \mathcal{S}^I \hookrightarrow \mathcal{P}^I)$ and all the functors on the right are left exact. So what has to be shown is that if $P \in \mathcal{P}^I$ satisfies $P^{\#^I} = 0$ (i.e., $P_i^\# = 0$ for all $i \in I$, whence $P_i^+ = 0$) then $\underset{\rightarrow}{P}^\# = (\varinjlim P)^\# = 0$. But if $\xi \in \underset{\rightarrow}{P}(U)$ is represented by $\xi \in P_i(U)$ (cf. I. Proposition (1. 3) !) then $\xi \longrightarrow 0$ in $\prod P_i(U_\alpha)$ for some $\{U_\alpha \longrightarrow U\} \in$ Cov T since $P_i^+ = 0$ and therefore

$\underset{\to}{\xi} \longrightarrow 0$ in $\prod \underset{\to}{P}(U_\alpha)$.

DEFINITION (5.1). Let T be a topology. T is <u>noetherian</u> iff. every $\{U_\alpha \longrightarrow U\} \in \mathrm{Cov}\, T$ has a finite subcover, i.e., a finite subfamily of $\{U_\alpha \longrightarrow U\}$ is in Cov T.

Note that this is quite rare.

PROPOSITION (5.2). Let T be a noetherian topology, and let T^f be the following topology: Cat T^f = Cat T, Cov T^f = set of families $\{U_\alpha \longrightarrow U\} \in \mathrm{Cov}\, T$ which are finite. Denote by $\mathcal{S}$, $\mathcal{S}^f$ the categories of sheaves on T, T^f respectively. Then $\mathcal{S} = \mathcal{S}^f$.

<u>Proof</u>: Obviously $\mathcal{S} \subset \mathcal{S}^f$. Let $F \in \mathcal{S}^f$. We have to show that if $\{U_\alpha \longrightarrow U\} \in \mathrm{Cov}\, T$ then $F(U) \xrightarrow{\sim} \ker\,(\prod F(U_\alpha) \rightrightarrows F(U_\alpha \underset{U}{\times} U_\beta))$. Let $\{U'_\nu \longrightarrow U\}$ be a finite subcover of $\{U_\alpha \longrightarrow U\}$. Then $F(U) \xrightarrow{\sim} \ker\,(\prod F(U'_\nu) \rightrightarrows \quad)$. Since this map factors through the canonical map $\ker\,(\prod F(U_\alpha) \rightrightarrows \quad) \longrightarrow \ker\,(\prod F(U'_\nu) \rightrightarrows \quad)$ it follows $F(U) \hookrightarrow \ker\,(\prod F(U_\alpha) \rightrightarrows \quad)$, i.e., F satisfies (+) (cf. (1.3)). Hence by (1.5) $\ker\,(\prod F(U_\alpha) \rightrightarrows \quad) \hookrightarrow \ker\,(\prod F(U'_\nu) \rightrightarrows \quad)$ and the result follows.

PROPOSITION (5.3). Suppose T noetherian and I a category satisfying (L1, 2). Let $F \in \mathcal{S}^I$. Then

(i) $\underset{\to}{\text{p. lim}}$ F is a sheaf.

(ii) If each F_i ($i \in I$) is flask in T^f then $\underset{\to}{\lim}$ F is flask in T^f, where T^f is as in (5.2).

Proof: By (5.2) we may assume $T = T^f$. Write $\underrightarrow{F} = p.\ \varinjlim F$ and let $\{U_\alpha \longrightarrow U\} \in \mathrm{Cov}\, T$ be a (finite) family. Now it is trivial that $\varinjlim$ commutes with direct sums. Therefore

$$\prod_{(\alpha_0, \dots, \alpha_n)} \underrightarrow{F}(U_{\alpha_0} \underset{U}{\times} \cdots \times U_{\alpha_n}) = \bigoplus_{(\alpha_0, \dots, \alpha_n)} \underrightarrow{F}(U_{\alpha_0} \underset{U}{\times} \cdots \underset{U}{\times} U_{\alpha_n})$$

$$= \varinjlim_{I} \left(\bigoplus F(U_{\alpha_0} \underset{U}{\times} \cdots \underset{U}{\times} U_{\alpha_n}) \right)$$

$$= \varinjlim_{I} \left(\prod_{(\alpha)} F(U_{\alpha_0} \underset{U}{\times} \cdots \underset{U}{\times} U_{\alpha_n}) \right) .$$

Since $\varinjlim_I$ is exact by I. (1.4) the proposition follows immediately.

COROLLARY (5.4). Let T be a noetherian topology, and I a category satisfying (L1, 2). The functors $H^q(T, U;\ )$ commute with $\varinjlim_I$.

Note that given $F : I \longrightarrow \mathcal{S}$ we get functors $h^q: I \longrightarrow (Ab)$ by $h^q(i) = H^q(T, U; F_i)$. Since by (5.2) we may assume $T = T^f$ and since $H^q(T, U;\ )$ may be computed with flask resolutions (cf. (4.4)) we are done by (5.3) if we can find a resolution $0 \longrightarrow F \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots$ of the functor F such that F^q_i is flask all $q > 0$, $i \in I$. In fact there exist resolutions with F^q_i injective. This can be seen without calculation as follows: $\mathcal{P}^I$ and $\mathcal{S}^I$ have all the axioms that $\mathcal{P}$, $\mathcal{S}$ do, so $\mathcal{S}^I$ has enough injectives and we take for F^q an injective resolution of F. If J is another category and $J \xrightarrow{\phi} I$ we get functors

$$\mathcal{P}^J \underset{\phi_p}{\overset{\phi^p}{\rightleftarrows}} \mathcal{P}^I \quad ,$$

adjoint, which are constructed as with preschemes (cf. I, Section 2) (so $\phi^p(P) = P \circ \phi$). In particular if $J = \{i\}$ is the discrete category, $i \in I$ and $\phi : \{i\} \longrightarrow I$ is the inclusion and if P is a presheaf = element of $\mathcal{P}^{\{i\}}$ then

$$(\phi_p P)_j = \bigoplus_{\mathrm{Hom}_I(i, j)} P \quad .$$

Similarly, we get

$$\mathcal{S}^J \underset{\phi_s}{\overset{\phi^s}{\rightleftarrows}} \mathcal{S}^I \quad ,$$

with $\phi^s(F) = F \circ \phi$ ($F \in \mathcal{S}^I$), and with $\phi_s = \#^I \circ \phi_p \circ i^J$. Therefore if $s \in \mathcal{S} = \mathcal{S}^{\{i\}}$ we have

$$\phi_s(S)_j = (\text{presheaf} \bigoplus_{\mathrm{Hom}_I(i,j)} S)^{\#} = \bigoplus_{\mathrm{Hom}_I(i,j)} S \quad .$$

Since AB5 holds in $\mathcal{S}$ it follows that $\phi_s : \mathcal{S} \longrightarrow \mathcal{S}^I$ is exact and therefore that ϕ^s carries injectives into injectives. Since $\phi^s F = F_i$

in this case, we are done.

A special case of (5.3), (5.4) is

COROLLARY (5.5). If T is noetherian then

(i) "presheaf $\oplus_\nu F_\nu$" is a sheaf.

(ii) $H^q(T, U;\quad)$ commutes with $\oplus$.

CHAPTER III. The étale Grothendieck topology for schemes.

Section 1. Definition. Let X be a prescheme and consider the following topologies:

(f) Cat T_X^f = Category of preschemes Y/X étale, separated and finitely presented.

Cov T_X^f = Finite families $\{U_\alpha \longrightarrow U\}$ of maps which are surjective, i. e, such that U is covered by the union of the images of the U_α s .

(0) Cat T_X^0 = Preschemes Y/X étale, separated, finitely presented.

Cov T_X^0 = Arbitrary surjective families $\{U_\alpha \longrightarrow U\}$.

(1) Cat T_X^1 = Preschemes Y/X étale, separated.

Cov T_X^1 = Arbitrary surjective families $\{U_\alpha \longrightarrow U\}$.

(2) Cat T_X^2 = Preschemes Y/X étale.

Cov T_X^2 = Arbitrary surjective families $\{U_\alpha \longrightarrow U\}$.

We refer to these as cases (f), (0), (1), (2) respectively, and we are primarily interested in cases (f) and (1). The definition of étale used is that of SGA IX, so in particular étale implies locally of finite presentation. Notice that the maps in Cat T_X have in each case the properties required of the structure maps, and that Cat T_X is closed under fibred products (in the category of preschemes).

We have inclusions Cat $T_X^f \xrightarrow{\alpha}$ Cat $T_X^0 \xrightarrow{\beta}$ Cat $T_X^1 \xrightarrow{\gamma}$ Cat T_X^2 which are obviously morphisms of topologies in the sense of II (4.5). Denoting by $\mathcal{S}^z$ the category of abelian sheaves on T_X^z ($z = f, 0, 1, 2$), we get functors $\mathcal{S}^2 \xrightarrow{\gamma^s} \mathcal{S}^1 \xrightarrow{\beta^s} \mathcal{S}^0 \xrightarrow{\alpha^s} \mathcal{S}^f$ where α^s is an inclusion.

THEOREM (1.1).

(i) If X is quasicompact then α^s is the identity.

(ii) If X is quasiseparated then β^s is an equivalence of categories.

(iii) γ^s is an equivalence of categories.

Proof: (i) follows from II (5.2) and the following

LEMMA (1.2). If X is quasicompact then T_X^0 is noetherian.

Proof of (1.2): Let $\{Y_\alpha \longrightarrow Y\} \in \mathrm{Cov}\, T_X^0$. We have to show that already a finite number of images cover Y. Since Y/X is finitely presented, it is quasicompact. Hence Y is quasicompact. Also Y_α/Y is étale, separated finitely presented. So we may assume Y = X. Now one sees easily that an étale morphism is open, using the fact that this is true if X is noetherian (cf. SGA IV, remark following Theorem 6.6). Hence the image of each Y_α is open. Since X is quasicompact, we are done.

To prove (ii), (iii) we need the following

LEMMA (1.3). Let $i : T' \subset T$ be topologies (i a morphism) and suppose

(1) Cat T' is a full subcategory of Cat T.

(2) If $\{U_\alpha \longrightarrow U\} \in \text{Cov } T$, and U_α, $U \in \text{Cat } T'$ for all α then $\{U_\alpha \longrightarrow U\} \in \text{Cov } T'$.

(3) Given $U \in \text{Cat } T$ $\exists$ $\{U_\alpha \longrightarrow U\} \in \text{Cov } T$ with $U_\alpha \in \text{Cat } T'$, all α.

Then $i^s : \mathcal{S}_T \longrightarrow \mathcal{S}_{T'}$ is an equivalence of categories.

Assuming the lemma, we are reduced to checking (1), (2), (3) for β, γ with X quasiseparated in case β. (1), (2) are trivial. For (3), recall that if Y/X is locally finitely presented then for every $y \in Y$ there is an open $V \subset Y$ containing y with V/X locally finitely presented and separated. For, by definition $\exists$ $V \subset Y$ and $U \subset X$ affine with V/U finitely presented (and certainly separated). Since U/X is obviously locally finitely presented and separated, so is V/X. Hence if $Y \in \text{Cat } T^2_X$ we can find a Zariski open covering in $\text{Cat } T^1_X$. Finally, if X is quasiseparated, and V, U are as above then $U \subset X$ is quasicompact, hence also finitely presented and so V/X is finitely presented and separated. Hence the Zariski open covering is in $\text{Cat } T^0_X$ in this case.

Proof of (1.3): Since the functors $i^s : \mathcal{S}_T \longrightarrow \mathcal{S}_{T'}$ and $i_s : \mathcal{S}_{T'} \longrightarrow \mathcal{S}_T$ are adjoint, there are canonical morphisms of functors $\phi : i_s i^s \longrightarrow {}^{id}\mathcal{S}_T$ and $\psi : {}^{id}\mathcal{S}_{T'} \longrightarrow i^s i_s$. We have to show these are isomorphisms. To show ψ is an isomorphism we need to show for $F \in \mathcal{S}_{T'}$, $V \in \text{Cat } T'$ that $F(V) \xrightarrow{\sim} i_s F(V) = i^s i_s F(V)$. Now clearly $F(V) \xrightarrow{\sim} i_p F(V)$ since (V, id_V) is an initial object in the category I^i_V (cf. I, (2.2)). Applying the assumptions of the lemma, we find that the coverings from T' of V are initial (final in the dual category) in the category of coverings of V in T, and since F is a sheaf on T' we find $i_p F(V) \xrightarrow{\sim} (i_p F(V))^{\#} = i_s F(V)$. So we are done.

To show ϕ is an isomorphism we want $i_s i^s G(U) \xrightarrow{\sim} G(U)$ for $U \in \text{Cat } T$, $G \in \mathcal{S}_T$.

Case (1): $U \in \text{Cat } T'$. Then (U, id_U) is initial in I^i_U, so

$$i_p i^s G(U) = \varinjlim_{(V,\phi)\in I^i_U} i^s G(V) = \varinjlim G(V) \xrightarrow{\sim} G(U) \quad .$$

Again $(i_p i^s G(U))^{\#} = i_s i^s G(U) \xrightarrow{\sim} G(U)$ because we only have to look at coverings from T' and we have isomorphism there.

Case (2): $U \in \mathrm{Cat}\, T$ arbitrary. Applying (3), we may choose $\{U_\alpha \longrightarrow U\} \in \mathrm{Cov}\, T$ with $U_\alpha \in \mathrm{Cat}\, T'$. Choose also for each pair α, β of indices $\{V^\nu_{\alpha\beta} \longrightarrow U_\alpha \underset{U}{\times} U_\beta\} \in \mathrm{Cov}\, T$ with $V^\nu_{\alpha\beta} \in \mathrm{Cat}\, T'$. Then for any $S \in \mathcal{S}_T$ we have

$$\prod_{\alpha,\beta} S(U_\alpha \underset{U}{\times} U_\beta) \hookrightarrow \prod_{\alpha,\beta,\nu} S(V^\nu_{\alpha,\beta}) \quad .$$

Hence in the diagram below the rows are exact. Since by case (1) the two right vertical arrows are isomorphisms (the arrows induced by ϕ), so is the left one. This completes the proof of the lemma.

$$\begin{array}{ccccccc}
0 \longrightarrow & i_s i^s G(U) & \longrightarrow & \prod i_s i^s G(U_\alpha) & \rightrightarrows & \prod i_s i^s G(V^\nu_{\alpha\beta}) \\
& \downarrow & & \downarrow \wr & & \downarrow \wr \\
0 \longrightarrow & G(U) & \longrightarrow & \prod G(U_\alpha) & \rightrightarrows & \prod G(V^\nu_{\alpha\beta})
\end{array}$$

PROPOSITION (1.4). The coverings in the cases (f), (0), (1), (2) are universal effective epimorphisms in the category of preschemes.

We omit the proof. One reduces easily to the case $\coprod U_\alpha \longrightarrow U$ faithfully flat and quasicompact (cf. SGA VIII Corollary 5.3).

Let $\pi : X \longrightarrow Y$ be a morphism of preschemes. π induces a morphism of topologies $f(\pi) = f : T_Y^z \longrightarrow T_X^z$ ($z = f, 0, 1, 2$) by $f(U) = U \times_Y X$ for $U \in \mathrm{Cat}\, T_Y^z$.

NOTATION (I. 5). Assuming z known and fixed we write

$\mathcal{S}_X = \mathcal{S}_{T_X^z} =$ category of abelian sheaves on T_X^z

$\mathcal{P}_X = \mathcal{P}_{T_X^z} =$ category of abelian presheaves on T_X^z

$\pi_* = f^s$

$\pi^* = f_s$

$H^q(X; F) = H^q(T_X^z, X; F) \approx H^q(T_X^z; F) \qquad (F \in \mathcal{S}_X)$.

Since f preserves final objects and fibred products we have by (II. (4.14))

COROLLARY (I. 6). $\pi^* = f_s$ is an exact functor.

Suppose $P = \mathrm{spec}\, \bar{K}$ where $\bar{K}$ is a separably algebraically closed field. Then T_P^z is equivalent with the canonical topology (cf. I. (0. 3)) on the category of sets (resp. finite sets if $z = 0$, resp. finite sets, finite covering families if z = f). In any case, $\mathcal{S}_P$ is equivalent with (Ab), the equivalence being given by $G \rightsquigarrow G(P)$, $G \in \mathcal{S}_P$.

DEFINITION (I. 7). Let $\epsilon : P \longrightarrow X$ be a geometric point of X, i. e., $P = \operatorname{spec} \bar{k}$ with $\bar{k}$ separably algebraically closed, and let $F \in \mathcal{S}_X$. The <u>stalk of F at</u> P is the abelian group $F_P = \epsilon^* F(P)$.

The functor $F \rightsquigarrow F_P$ is exact by (I. 6) and the above remarks. One sees easily, moreover, that if $f : T_X \longrightarrow T_P$ is the morphism associated to ϵ then

$$\epsilon^* F(P) = f_s F(P) = f_p F(P), \text{ i.e., } F_P = \varinjlim_{(V,\phi) \in I_P^f} F(V) \quad .$$

Since $(X, \mathrm{id}_P) \in I_P^f$ we get a canonical homomorphism $F(X) \longrightarrow F_P$.

PROPOSITION (I. 8). Suppose X quasicompact in case (f), and let $F \in \mathcal{S}_X$, $\xi \in F(X)$. Then $\xi = 0$ iff. the image of ξ in F_P is zero for every geometric point P.

Proof: $\Longrightarrow$ is trivial. So suppose $\xi \rightsquigarrow 0$ in every F_P, and let $x \in X$. Choose $\epsilon : P \longrightarrow X$ lying over x. Since I_P^f satisfies $(L1, 2, 3)^0$ (cf. proof of II. (4. 14)) we may argue as follows: There is some $U/X \in \operatorname{Cat} T_X$, and a map $\phi : P \longrightarrow U$ commuting with ϵ so that $\xi \rightsquigarrow 0$ in F(U). Then the image of U on X contains x. Since this can be done for each x, we can find $\{U_\alpha \longrightarrow U\} \in \operatorname{Cov} T_X$ (finite in case (f)) with $\xi \rightsquigarrow 0$ in $\prod F(U_\alpha)$. Therefore $\xi = 0$.

PROPOSITION (1.9). Let F be a presheaf on X, $P \longrightarrow X$ as above a geometric point. Then $f_p F(P) \approx f_p F^{\#}(P) \approx F^{\#}_P$, where the isomorphism is induced by the canonical map $F \longrightarrow F^{\#}$.

The proof is essentially obvious. For instance, to show the map surjective, let $\underline{\xi} \in f_p F^{\#}(P)$ be represented by $\xi \in F^{\#}(V)$ for some $\phi : P \longrightarrow V$ commuting with ϵ. Then there is a $\{V_\alpha \longrightarrow V\} \in \text{Cov } T$, and $\xi' \in \prod F^+(V_\alpha)$ representing ξ. Since $\coprod V_\alpha \longrightarrow V$ is surjective we can lift ϕ to some map $\phi_\alpha : P \longrightarrow V_\alpha$. Let ξ'_α be the αth component of ξ', and choose $\{W_\beta \longrightarrow V_\alpha\} \in \text{Cov } T$ and $\xi'' \in \prod F(W_\beta)$ representing ξ'_α. Lift ϕ_α to some $\phi_\beta : P \longrightarrow W_\beta$ and let ξ'' be the β th component of ξ''. Obviously the image $\underline{\xi}''$ of ξ''_β in $f_p F(P)$ (obtained with respect to $(W_\beta, \phi_\beta) \in I^f_P$) is mapped onto $\underline{\xi}$.

Section 2. <u>Relations with a closed subscheme.</u> Throughout this section we assume we are in case (1). Let $i : X \longrightarrow Y$ be a closed subscheme and let $F \in \mathcal{S}_Y$.

DEFINITION (2.1). F is <u>zero outside</u> X iff. $F(U) = 0$ whenever $U \times_Y X = \emptyset$.

Clearly i_* carries $\mathcal{S}_X$ into the full subcategory $\bar{\mathcal{S}}_Y$ of $\mathcal{S}_Y$ consisting of sheaves zero outside X.

THEOREM (2.2). The functor $\mathcal{S}_X \longrightarrow \bar{\mathcal{S}}_Y$ induced by i_* is an equivalence of categories.

Proof: Let $\epsilon : T_X' \subset T_X$ be the full subtopology (i.e., all maps and coverings) whose objects are preschemes $\bar{V}/X$ where $\bar{V} = V \times_Y X$ for some $V \in \text{Cat}\, T_Y$. We claim T_X' satisfies the conditions of Lemma (1.3) and hence that we may replace T_X by T_X'. Condition (iii) is the only nontrivial one. Let $Z/X \in T_X$, $z \in Z$. By definition of étale, we can find $z \in \bar{V} \subset Z$, $\bar{U} \subset X$ affine with $\bar{V} \longrightarrow \bar{U}$ and affines $\bar{V}_0/\bar{U}_0$ of finite type over spec $\mathbb{Z}$ and maps $\bar{U} \longrightarrow \bar{U}_0$, $\bar{V} \xrightarrow{\sim} \bar{V}_0 \times_U U_0$. We may assume $\bar{U} = U \times X$ for some affine $U \subset Y$ and then we can choose $U \longrightarrow U_0$ and $\bar{U}_0 \longrightarrow U_0$ closed, with U_0 of finite type over spec $\mathbb{Z}$. Let $z_0 \in \bar{V}_0$ be the image of z. We are reduced to the noetherian case, and it follows easily from SGA I, Theorem 7.6 and Proposition 4.5.

Replace T_X by T_X' and let $f : T_Y \longrightarrow T_X'$ be the morphism $f(U) = U \times_Y X$. We want to show f^s induces an equivalence $f^0 : \mathcal{S}_X \longrightarrow \bar{\mathcal{S}}_Y$. The adjoint functor $f_0 : \bar{\mathcal{S}}_Y \longrightarrow \mathcal{S}_X$ is obviously obtained from f_s by restricting to $\bar{\mathcal{S}}_Y$, so we have morphisms $\phi : f_0 f^0 \longrightarrow \text{id}_{\mathcal{S}_X}$, $\psi : \text{id}_{\bar{\mathcal{S}}_Y} \longrightarrow f^0 f_0$ which we want to show are isomorphisms. Actually, it will appear that $f_p | \bar{\mathcal{S}}_Y$ is the adjoint since $f_p F$ is a sheaf for $F \in \bar{\mathcal{S}}_Y$.

Let $\bar{V} \in T_X'$, $\bar{V} = V \times_Y X$ with $V \in T_Y$ and consider the category

$$I_{\bar{V}}^f = \{(U, \phi) \mid U \in T_Y \;\; , \;\; \phi : U \times_Y X \longleftarrow \bar{V}\}.$$

We claim that the (U, ϕ) s in $I\frac{f}{V}$ with ϕ an isomorphism form an initial subcategory. In fact, if (U, ϕ) is arbitrary then $(U, \phi) \longleftarrow (V \times_Y U, \Gamma_\phi)$ (identifying $V \times_Y U \times_Y X$ with $\bar{V} \times_X (U \times_Y X)$). Write $\bar{U} = U \times_Y X$. The graph $\Gamma_\phi : \bar{V} \longrightarrow \bar{V} \times_X \bar{U}$ is open and closed (SGA IX, Corollary 1.6 and above), hence we may write $\bar{V} \times_X \bar{U} = \Gamma \amalg \Lambda$ where $\bar{V} \xrightarrow{\sim} \Gamma$. In the map $\bar{V} \times_X \bar{U} \longrightarrow V \times_Y U$ (which is a closed immersion), Λ is mapped onto a closed subset. Set $W = U \times_Y V - \Lambda$. Then $W \in T_Y$ and we get a map $(U, \phi) \longleftarrow (W, \psi)$ with ψ an isomorphism.

Next, we claim that for $F \in \bar{\mathcal{S}}_Y$, $V \in T_Y$, $F(V) \longrightarrow f_p F(\bar{V})$ is an isomorphism, where $\bar{V} = V \times_Y X$ and the map is given by $(V, id_{\bar{V}}) \in I\frac{f}{V}$. This is trivial if $V \times_Y X = \emptyset$. Since $(L1, 2, 3)^0$ hold in $I\frac{f}{V}$ (cf. proof of II (4.14)) and by the above discussion, we may replace $I\frac{f}{V}$ by the category of pairs (U, ϕ) in which ϕ is an isomorphism, and then it suffices to show $F(U) \xrightarrow{\sim} F(U')$ for $(U, \phi) \longleftarrow (U', \phi')$ (and ϕ, ϕ' isomorphisms). View $\bar{U} = U \times_Y X$ as a closed subscheme of U. Since ϕ is an isomorphism, U' covers $\bar{U}$ and therefore covers some open set containing it. Hence we may separate the two cases

(1): $U' \subset U$ is an open subset.

(2): $U' \longrightarrow U$ surjective, i.e., $\{U' \longrightarrow U\} \in \mathrm{Cov}\, T_Y$.

Case (1): Set $U'' = U - \bar{U}$, so $U' \amalg U'' \longrightarrow U$ is surjective. Then $U'' \times_Y X = \emptyset$, hence $F(U'') = 0$ and we are reduced to case (2).

Case (2): Consider

$$U \longleftarrow U' \;\underset{p_2}{\overset{p_1}{\leftleftarrows}}\; U' \underset{U}{\times} U' \quad ,$$

and let $\Delta \subset U' \underset{U}{\times} U'$ be the image of the diagonal. Δ is open and closed in $U' \underset{U}{\times} U'$, so we can write $U' \underset{U}{\times} U' = \Delta \amalg \Lambda$. Since ϕ and ϕ' are isomorphisms, $U' \underset{Y}{\times} X \xrightarrow{\sim} U \underset{Y}{\times} X$. Therefore $U' \underset{U}{\times} U' \underset{Y}{\times} X \xrightarrow{\sim} U' \underset{Y}{\times} X$, and since certainly $\Delta \underset{Y}{\times} X \xrightarrow{\sim} U' \underset{Y}{\times} X$ we find $\Lambda \underset{Y}{\times} X = \emptyset$. Hence $F(\Lambda) = 0$. But $p_1 = p_2$ on Δ. Therefore $F(U') \rightrightarrows F(U' \underset{U}{\times} U')$ is the zero map and since F is a sheaf, $F(U) \xrightarrow{\sim} F(U')$ as required.

Now it is clear that $f_p F$ is a sheaf. For, if $\{\bar{V}_\alpha \longrightarrow \bar{V}\} \in \text{Cov } T'_X$, $\bar{V}_\alpha = V_\alpha \underset{Y}{\times} X$, $\bar{V} = V \underset{Y}{\times} X$, we may replace $\bar{V}_\alpha$ by isomorphic elements of Cat T'_X, say $\bar{V}'_\alpha$ so that $\bar{V}'_\alpha \longrightarrow \bar{V}$ is induced by a map $V'_\alpha \longrightarrow V$. Since $\{V'_\alpha\}$ cover $\bar{V}$, they cover some open neighborhood of $\bar{V}$. Hence we may assume $\{\bar{V}_\alpha \longrightarrow \bar{V}\}$ is induced from a covering $\{V_\alpha \longrightarrow V\} \in \text{Cov } T_Y$. Since F is a sheaf we are done.

The fact that $\phi : f_0 f^0 \longrightarrow \text{id}_{\mathcal{S}_X}$ and $\psi : \text{id}_{\mathcal{S}_Y} \longrightarrow f^0 f_0$ are isomorphisms is now immediate.

By transitivity of equivalence we have:

COROLLARY (2. 3). The functor of Theorem (2. 2) is an equivalence of categories in case (2), and in case (0) if Y is quasiseparated, and in case (f) if Y is quasicompact and quasiseparated.

Let for the moment $\mathcal{A} \xleftarrow{f} \mathcal{B}$ be any left exact functor, $\mathcal{A}$, $\mathcal{B}$ abelian categories. Construct a new category $\mathcal{C}$ as follows: Ob $\mathcal{C}$ = set of triples (B, A, ϕ) where $B \in \mathcal{B}$, $A \in \mathcal{A}$, $\phi \in \mathrm{Hom}(A, f(B))$. A map $(B, A, \phi) \xrightarrow{\xi} (B', A', \phi')$ is a pair $B \xrightarrow{\xi_B} B'$ and $A \xrightarrow{\xi_A} A'$ of maps such that

$$\begin{array}{ccc} A & \xrightarrow{\phi} & f(B) \\ {\scriptstyle \xi_A}\downarrow & & \downarrow{\scriptstyle f(\xi_B)} \\ A' & \xrightarrow{\phi'} & f(B') \end{array}$$

commutes. Using the fact that f is left exact one verifies easily that $\mathcal{C}$ is again an abelian category. $\mathcal{C}$ inherits most properties enjoyed by $\mathcal{A}$, $\mathcal{B}$.

There are various functors relating $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, notably the six

$$\mathcal{A} \;\begin{array}{c} \xleftarrow{\quad i^* \quad} \\ \xrightarrow{\quad i_* \quad} \\ \xleftarrow{\quad i^! \quad} \end{array}\; \mathcal{C} \;\begin{array}{c} \xleftarrow{\quad j_! \quad} \\ \xrightarrow{\quad j^* \quad} \\ \xleftarrow{\quad j_* \quad} \end{array}\; \mathcal{B}$$

defined as follows:

$$i^*: A \leftsquigarrow (B, A, \phi) \qquad\qquad j_! : (B, 0, 0) \leftsquigarrow B$$

$$i_*: A \rightsquigarrow (0, A, 0) \qquad ; \qquad j^* : (B, A, \phi) \rightsquigarrow B$$

$$i^!: \ker \phi \leftsquigarrow (B, A, \phi) \qquad\qquad j_* : (B, f(B), id) \leftsquigarrow B$$

We insert for reference a list of obvious properties:

(i) A given functor is left adjoint to the one below it.

(ii) i^*, i_*, j^*, $j_!$ are exact, j_*, $i^!$ are left exact.

(iii) $i^* j_* = f$, $i^* j_! = i^! j_! = i^! j_* = j^* i_* = 0$.

(iv) i_*, j_* are fully faithful and for $C \in \mathcal{C}$, $j^* C = 0$ iff $C \approx i_* A$, some $A \in \mathcal{A}$.

PROPOSITION (2.4). Let

$$\mathcal{A} \;\underset{i_*}{\overset{i^*}{\leftrightarrows}}\; \bar{\mathcal{C}} \;\underset{j_*}{\overset{j^*}{\rightleftarrows}}\; \mathcal{B}$$

be abelian categories and functors. Suppose the functors satisfy the following conditions:

(i) i^* (resp. j^*) is left adjoint to i_* (resp. j_*).

(ii) i^*, j^* are exact.

(iii) i_*, j_* are fully faithful.

(iv) For $C \in \bar{\mathcal{C}}$, $j^* C = 0 \Longleftrightarrow C = i_* A$, some $A \in \mathcal{A}$.

Set $f = i^* j_*$, and denote by $\mathcal{C}$ the category of triples $(B, A, \emptyset)$ as above. Then $\bar{\mathcal{C}}$ is equivalent with $\mathcal{C}$.

To be sure, the equivalence 'preserves' the given functors. Since j_* has a left adjoint, it is left exact and so f is left exact.

Proof: Note first that for $C \in \bar{\mathcal{C}}$, $C = 0$ iff. $i^* C = 0$ and $j^* C = 0$. For, say $j^* C = 0$, when $C \approx i_* A$ some $A \in \mathcal{A}$. Since i_* is fully faithful, the canonical map $i^* i_* A \longrightarrow A$ deduced from (i) is an isomorphism. So if also $i^* C \approx i^* i_* A = 0$ then $A = 0$, hence $C = 0$. Since i^*, j^* are exact we conclude that a map $\epsilon : C \longrightarrow C'$ in $\bar{\mathcal{C}}$ is an isomorphism iff. $i^*(\epsilon)$, $j^*(\epsilon)$ are isomorphisms.

Let $C \in \bar{\mathcal{C}}$ be arbitrary. The adjointness properties (i) induce a commutative diagram

$$\begin{array}{ccc} i_* i^* C & \longleftarrow & C \\ \downarrow & & \downarrow \\ i_* i^* j_* j^* C & \longleftarrow & j_* j^* C \end{array}$$

and hence a map $C \xrightarrow{\epsilon} (i_* i^* C) \underset{(i_* i^* j_* j^* C)}{\times} (j_* j^* C)$.

We claim ϵ is an isomorphism, and we need to show $i^*(\epsilon)$, $j^*(\epsilon)$ are isomorphisms. Both functors i^*, j^* are exact by assumption and therefore commute with fibered products. Since i_* is fully faithful, $i^*C \longrightarrow i^*i_*i^*C$ is an isomorphism for all $C \in \mathcal{C}$, hence the horizontal arrows of the diagram become isomorphisms upon applying i^*, so the square certainly becomes cartesian. Applying j^* to the diagram we get zero on the left side by (iv), and $j^*C \xrightarrow{\sim} j^*j_*j^*C$ because j_* is fully faithful, so again the square becomes cartesian and we are done.

This argument applies of course also to the category $\mathcal{C}$ of triples, so we have $(B, A, \emptyset) \xrightarrow{\sim} (0, A, 0) \underset{(0, f(B), 0)}{\times} (B, f(B), \mathrm{id})$.

Therefore define a functor $F : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ by $C \rightsquigarrow (j^*C,\ i^*C, \emptyset)$ where $\emptyset : i^*C \longrightarrow i^*j_*j^*C = f(j^*C)$ is induced by the map $C \longrightarrow j_*j^*C$, and define $G : \mathcal{C} \longrightarrow \bar{\mathcal{C}}$ by $(B, A, \emptyset) \rightsquigarrow i_*A \underset{i_*i^*j_*B}{\times} j_*B$. It is immediate that one obtains an equivalence of categories.

COROLLARY (2.5). Let $i : X \longrightarrow Y$ be a closed subscheme and set $U = Y - X$, $j : U \hookrightarrow Y$ the map. The category $\mathcal{S}_Y$ of sheaves on Y is equivalent with the category of triples $(B, A, \emptyset)$ where $B \in \mathcal{S}_U$, $A \in \mathcal{S}_X$ and $\emptyset : A \longrightarrow i^*j_*B$.

For, if we set $\mathcal{A} = \mathcal{S}_X$, $\mathcal{B} = \mathcal{S}_U$, $\bar{\mathcal{C}} = \mathcal{S}_Y$ in the above proposition then condition (i) is satisfied by definition. For (ii) see (1.6) and for (iv) see (2.2). Condition (iii) for i_* follows also from (2.2), and for j_* we need only show $B \xrightarrow{\sim} j^*j_*B$, $B \in \mathcal{S}_U$. This is clear (see for example I.(2.8)).

In this context the functor $i^!$ associates with a sheaf on Y the subsheaf of sections with support on X, and $j_!$ is the extension of a sheaf on U by zero. This interpretation greatly facilitates the calculation of the usual exact cohomology sequences for closed subsets, for instance we have immediately

EXACT SEQUENCE (2.6). For $C \in \mathcal{S}_Y$

$$0 \longrightarrow j_! j^* C \longrightarrow C \longrightarrow i_* i^* C \longrightarrow 0$$

which, read in the category $\mathcal{C}$ is

$$0 \longrightarrow (B, 0, 0) \longrightarrow (B, A, \emptyset) \longrightarrow (0, A, 0) \longrightarrow 0 \quad .$$

We also have a left exact sequence

$$(2.7) \qquad 0 \longrightarrow i_* i^! C \longrightarrow C \longrightarrow j_* j^* C \quad , \qquad C \in \mathcal{C} .$$

If $C \in \mathcal{C}$ is an injective then the last arrow is in fact surjective (say we work in $\mathcal{C}$, assuming $\mathcal{A}$, $\mathcal{B}$ (hence $\mathcal{C}$) have enough injectives). For, if C is injective, so are $i_* i^! C$ and $j_* j^* C$. Hence $C \approx i_* i^! C \times C'$ with C' also injective and $j_* j^* C \approx j_* j^* C'$. Therefore $j_* j^* C \approx C' \times C''$. But since j_* is fully faithful, $j^* C' \longrightarrow j^* j_* j^* C \approx j^* C' \times j^* C''$ is an isomorphism and so $j^* C'' = 0$. Since also

$$C' \times C'' \approx j_* j^* C \xrightarrow{\sim} j_* j^* j_* j^* C \approx j_* j^* C' \times j_* j^* C''$$

it follows that $C'' = 0$ and we are done.

Therefore we obtain an exact cohomology sequence

$$\cdots \longrightarrow R^q(i_* i^!) C \longrightarrow R^q \, \mathrm{id} C \longrightarrow R^q(j_* j^*) C \longrightarrow \cdots \quad .$$

We have $R^q \, \mathrm{id} = 0$, $R^q(i_* i^!) \approx i_* R^q i^!$, $R^q(j_* j^*) \approx (R^q j_*) j^*$, and so we find

$$i_* R^1 i^! C \approx \mathrm{coker} \; (C \longrightarrow j_* j^* C)$$

(2.8.1) or

$$R^1 i^! (B, A, \emptyset) \approx \mathrm{coker} \; \emptyset$$

(2.8.q) $$i_* R^q i^! C \approx (R^{q-1} j_*) j^* C \quad , \quad q > 1$$

or, since $i^* i_* \approx \mathrm{id}_{\mathcal{a}}$ and $i^* R^q j_* \approx R^q i^* j_* = R^q f$

(2.8.q') $$R^q i^! (B, A, \emptyset) \approx R^{q-1} f (B) \quad , \quad q > 1 \quad .$$

Note that $R^q j_*$ $(q > 0)$ has always the form $(0, x, 0)$, i.e., in case of the closed subscheme $Y - X = U \xrightarrow{j} Y$, the sheaves $R^q j_* B$ are zero outside of X $(q > 0)$.

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a left exact functor ($\mathcal{D}$ another abelian category). Since for $C \in \mathcal{C}$ injective the sequence (2.7) is exact and consists of injectives, we get an exact sequence $0 \longrightarrow Fi_* i^! (C) \longrightarrow F(C) \longrightarrow Fj_* j^* (C) \longrightarrow 0$. Therefore for arbitrary $C \in \mathcal{C}$ we get

EXACT SEQUENCE (2.9).

$$\cdots \longrightarrow R^q (Fi_* i^!)\, C \longrightarrow R^q F\, C \longrightarrow R^q (Fj_* j^*)\, C \longrightarrow \cdots$$

NOTATION (2.10). For $X \xrightarrow{i} Y$ a closed subscheme, $U = X - Y$, $C \in \mathcal{S}_Y$, we write

$$H^0_X (Y, C) = H^0 (X, i^! C) = H^0 (Y, i_* i^! C) \quad ,$$

$$H^q_X (Y, \quad) = R^q (H^0 (X, i^! \quad)) \quad .$$

$H^0_X (Y, C)$ is the group of sections of C with support on X, and $H^q_X (Y, C)$ is the relative cohomology. Of course, there is a spectral sequence relating $H^q_X (Y, \quad)$ with $R^q i^!$. Taking into account the fact that j_* (i_*) is ~~exact~~ and carries injectives to injectives, (2.9) reads

RELATIVE COHOMOLOGY SEQUENCE (2.11).

$$\cdots \longrightarrow H^q_X (Y, C) \longrightarrow H^q (Y, C) \longrightarrow H^q (U, j^* C) \longrightarrow \cdots \quad .$$

This should be contrasted with the exact sequence arising from (2.6), viz.

$$\text{(2.12).} \qquad \cdots \longrightarrow H^q(Y,\, j_! j^* C) \longrightarrow H^q(Y, C) \longrightarrow H^q(X, i^* C) \longrightarrow \cdots$$

Since $j_!$ does not in general carry injectives into injectives we can <u>not</u> write $H^q(Y,\, j_! j^* \quad) = R^q(H^0(Y, j_! j^* \,.\,))$.

For the sake of completeness, we note that a left exact functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ induces functors $F_a : \mathcal{A} \longrightarrow \mathcal{D}$ and $F_b : \mathcal{B} \longrightarrow \mathcal{D}$, left exact, by $F_a = Fi_*$, $F_b = Fj_*$, and a morphism $F_f : F_b \longrightarrow F_a f$ by $j_* B \longrightarrow i_* i^* j_* B = i_* f(B)$. Since F is left exact, it commutes with products and therefore we have for $C = (B, A, \emptyset) \in \mathcal{C}$ a cartesian diagram

$$\begin{array}{ccc} & F(C) & \\ \swarrow & & \searrow \\ Fi_*(A) & & Fj_*(B) \\ \searrow & & \swarrow \\ & Fi_* i^* j_*(B) & \end{array} \quad = \quad \begin{array}{ccc} & F(C) & \\ \swarrow & & \searrow \\ F_a(A) & & F_b(B) \\ {\scriptstyle F_a(\emptyset)}\searrow & & \swarrow{\scriptstyle F_f(B)} \\ & F_a(f(B)) & \end{array}$$

Hence F can be recovered if $F_a \xrightarrow{F_f} F_b f$ are known. Thus for instance, if we are in the case of a closed subscheme $X \xrightarrow{i} Y$, $U = Y - X$, the section functors Γ_V (cf. II (2.1)) for $V \subset T_Y$ are determined by the functors $\Gamma_{V \times_Y X}$ on $\mathcal{S}_X$ and $\Gamma_{V \times_Y U}$ on $\mathcal{S}_U$ and by the canonical morphism

$$\Gamma_{V \times_Y U} \longrightarrow \Gamma_{V \times_Y X}\; i^* j_* \quad .$$

Section 3. Passage to the limit. Let J be a category. Recall the following definition: A pseudofunctor $C : J \longrightarrow (\mathrm{Cat})$ is

(a) a map $\mathrm{Ob}\, J \longrightarrow \mathrm{Ob}\, \mathrm{Cat}$; $j \rightsquigarrow C_j$.

(b) a map $F\ell\, J \longrightarrow F\ell\, \mathrm{Cat}$; $i \xrightarrow{f} j \rightsquigarrow C_i \xrightarrow{f_c} C_j$.

(c) For $i \xrightarrow{g} j \xrightarrow{f} k$ an isomorphism of functors

$c_{f,g} : f_c . g_c \xrightarrow{\sim} (fg)_c$

such that $c_{f,id} = c_{id,f} = id$ and if $i \xrightarrow{h} j \xrightarrow{g} k \xrightarrow{f} \ell$ are three maps then

$$c_{f,gh} \circ (f * c_{g,h}) = c_{fg,h} \circ (c_{f,g} * h)$$

(notation as in Godement).

Let $D : J \longrightarrow (\mathrm{Cat})$ be another pseudofunctor. A morphism $F : C \longrightarrow D$ is

(a) for $j \in J$ a functor $F_j : C_j \longrightarrow D_j$.

(b) for $i \longrightarrow j$ an isomorphism of functors $F_f : f_d \circ F_i \xrightarrow{\sim} F_j \circ f_c$ such that for $i \xrightarrow{g} j \xrightarrow{f} k$ in J

$$F_{fg} = c_{f,g} F_f F_g d^{-1}_{f,g} \quad .$$

The pseudofunctors and morphisms $J \longrightarrow (\mathrm{Cat})$ form a category. Let $X \in (\mathrm{Cat})$, and define the constant pseudofunctor $\mathrm{const}_X = c_X : J \longrightarrow (\mathrm{Cat})$ by $(c_X)_i = X$, $f_{c_X} = \mathrm{id}_X$, $(c_X)_{f,g} = \mathrm{id}$. It is clear that a functor $X \longrightarrow Y$ induces canonically a morphism $c_X \longrightarrow c_Y$.

DEFINITION (3.1): Let $C : J \longrightarrow (\mathrm{Cat})$ be a pseudofunctor. We define a functor $\underset{\longrightarrow}{\lim}\, C : (\mathrm{Cat}) \longrightarrow (\mathrm{Sets})$ by

$$[\underset{\longrightarrow}{\lim}\, C](X) = \mathrm{Hom}_{\mathrm{psfnct}}(C, c_X)$$

with the obvious maps.

We assume J satisfies (L1, 2) of I Section 1 for simplicity (For a more general discussion the reader may consult SGA VI). The point is that $\underset{\longrightarrow}{\lim}\, C$ is representable, i.e., there exists a category $\underset{\longrightarrow}{\lim}\, C = \underset{\rightarrow}{C}$, and a morphism of pseudofunctors $C \xrightarrow{L} \mathrm{const}_{\underset{\rightarrow}{C}}$ such that by composition with L

$$\mathrm{Hom}_{\mathrm{Cat}}(\underset{\rightarrow}{C}, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{psfnct}}(C, c_X) = [\underset{\longrightarrow}{\lim}\, C](X).$$

$\underset{\rightarrow}{C}$ may be constructed as follows: Set $\mathrm{Ob}\, \underset{\rightarrow}{C} = \coprod_i \mathrm{Ob}\, C_i$. For $U, V \in \mathrm{Ob}\, \underset{\rightarrow}{C}$ we have to define $\mathrm{Hom}_{\underset{\rightarrow}{C}}(U, V)$. Say $U \in \mathrm{Ob}\, C_i$, $V \in \mathrm{Ob}\, C_j$ and denote by $i, j \backslash J$ the category of diagrams

$$\begin{array}{ll} i \searrow & \\ & k \\ j \nearrow & \end{array}$$

where a map

$$\left(\begin{array}{ll} i \searrow & \\ & k \\ j \nearrow & \end{array}\right) \longrightarrow \left(\begin{array}{ll} i \searrow & \\ & k' \\ j \nearrow & \end{array}\right)$$

is a map $k \longrightarrow k'$ such that the resulting triangles commute. $i, j \backslash J$ satisfies (1.1, 2, 3) (cf. I (1.7) which is not stated with sufficient generality). Define a functor $H(U, V) = H : i, j \backslash J \longrightarrow (\text{Sets})$ by

$$\begin{array}{ll} i \overset{g}{\searrow} & \\ & k \\ j \underset{g'}{\nearrow} & \end{array} \quad \underset{H}{\rightsquigarrow} \quad \operatorname{Hom}(g_c(U), g'_c(V))$$

and for

$$\begin{array}{ll} i \overset{g}{\searrow} & \\ & k \xrightarrow{f} \ell \\ j \underset{g'}{\nearrow} & \end{array} \quad \text{let} \quad \operatorname{Hom}(g_c(U), g'_c(V)) \xrightarrow{H(i,j\backslash f)} \operatorname{Hom}(fg_c(U), fg'_c(V))$$

be defined by the functor f_c and the morphisms $c_{f,g}$, $c_{f,g'}$. Specifically, for $\emptyset \in \operatorname{Hom}(g_c(U), g'_c(V))$, we have

$$H(i, j\backslash f) : \emptyset \rightsquigarrow c_{f,g'}(V) \circ f_c(\emptyset) \circ c^{-1}_{f,g}(U) .$$

This gives indeed a functor and so we set

$$\mathrm{Hom}_{\underrightarrow{C}}(U,V) = \underrightarrow{\lim}\, H = \underrightarrow{\lim}\, H(U,V) .$$

The composition of maps in $\underrightarrow{C}$ is self explanatory, and one gets functors $L_i : C_i \longrightarrow \underrightarrow{C}$ by the inclusion on Ob C_i and the fact that

$$\begin{matrix} i \\ i \end{matrix} \geq i \in i, i \backslash J .$$

Together with suitable isomorphisms L_f these give the morphisms $L : C \longrightarrow \mathrm{const}_{\underrightarrow{C}}$. We omit a few pages of tedious verification.

Similarly, we define a <u>pseudofunctor</u> $T : J \longrightarrow$ (Top) (cf. II (4.5)) to be a pseudofunctor of J into the underlying categories of (Top) such that the morphisms are in (Top), i.e., T consists of a map

$$\mathrm{Ob}\, J \longrightarrow \mathrm{Ob(Top)} \qquad (j \rightsquigarrow T_j) ,$$

a map

$$\mathrm{F}\ell\, J \longrightarrow \mathrm{F}\ell\,(\mathrm{Top}) \qquad (f \rightsquigarrow f_t)$$

and isomorphisms $t_{f,g} : f_t g_t \longrightarrow fg_t$ with the same axioms as above. A morphism $F : T \longrightarrow T'$ is a morphism of the underlying pseudofunctors Cat T, Cat T' : J $\longrightarrow$ (Cat) such that for $i \in J$, $F_i : \mathrm{Cat}\, T_i \longrightarrow \mathrm{Cat}\, T_i'$ is a morphism of topologies. For $X \in$ (Top), set

$$[\underset{\rightsquigarrow}{\lim}\, T](X) = \mathrm{Hom}_{\mathrm{psfnct}}(T, c_X) ,$$

so that $\underset{\rightsquigarrow}{\lim}\, T$ is a functor (Top) $\longrightarrow$ (Sets).

PROPOSITION (3.2): Let J be a category satisfying (L1, 2), $T : J \longrightarrow$ (Top) a pseudofunctor and suppose for all $j \in J$ that the covering families of T_j are finite families. Then $\underrightarrow{\lim}\ T$ is representable.

Proof: Set Cat $\underrightarrow{T}$ = $\underrightarrow{\lim}$ [Cat T] where $\underrightarrow{\lim}$ [Cat T] is as above. Let $\{U_\alpha \xrightarrow{\underline{\phi}_\alpha} U\}$ be a family of maps of Cat $\underrightarrow{T}$, say $U_\alpha \in \text{Cat } T_{i\alpha}$, $U \in \text{Cat } T_i$. Put $\{U_\alpha \longrightarrow U\}$ in Cov $\underrightarrow{T}$ if it is a finite family and if there exists $j \in J$, maps $i_\alpha \xrightarrow{g_k} j$ and $i \xrightarrow{g} j$ and elements $\phi_{\alpha j} \in \text{Hom}(g_\alpha(U_\alpha), g(U))$ representing $\underline{\phi}_\alpha$ such that $\{g_\alpha(U_\alpha) \xrightarrow{\phi_{\alpha j}} g(U)\} \in \text{Cov } T_j$. It is clear that with this definition axioms (1), (2) of I (0.1) are satisfied. We have to verify (3). So let $\{U_\alpha \xrightarrow{\underline{\phi}_\alpha} U\} \in \text{Cov } \underrightarrow{T}$, $V \xrightarrow{\underline{\psi}} U \in \text{Cat } \underrightarrow{T}$. Reflecting a moment on the construction of Cat $\underrightarrow{T}$ one sees that we may replace the objects and maps by an isomorphic situation so that U_α, U, $V \in \text{Cat } T_j$ for some fixed j, so that $\underline{\phi}_\alpha$, $\underline{\psi}$ are represented by maps ϕ_α, ψ in Cat T_j, and so that with these maps $\{U_\alpha \longrightarrow U\} \in \text{Cov } T_j$. Then since T_j is a topology $U_\alpha \times_U V$ exists and $\{U_\alpha \times_U V \xrightarrow{p_{\alpha 2}} V\} \in$ Cat $\underrightarrow{T}$ is in Cov $\underrightarrow{T}$. It remains to show that " $U_\alpha \times_U V$" is the product in Cat $\underrightarrow{T}$, i.e., that for $W \longrightarrow U \in \text{Cat } \underrightarrow{T}$, $\text{Hom}_U(W, U_\alpha \times_U V) \approx \text{Hom}_U(W, U_\alpha) \times \text{Hom}_U(W, V)$. For this purpose we may assume furthermore (fixing W and replacing by an isomorphic situation) that $W \in \text{Cat } T_j$ and $W \longrightarrow U$ is represented in Cat T_j. Thus the desired fact follows easily from the commutativity of $\underrightarrow{\lim}$ with products (cf. I Proposition (1.8)) and the following

LEMMA (3.3): Let $W \longrightarrow U \longleftarrow V \in \operatorname{Cat} T_j$. Then

$$\operatorname{Hom}_U^{\underrightarrow{T}}(W, V) \xleftarrow{\ \sim\ } \varinjlim_{j \xrightarrow{f} k \in j\backslash J} \operatorname{Hom}_{f_t(U)}(f_t(W), f_t(V))$$

where the maps $W \longrightarrow U \longleftarrow V$ in $\operatorname{Cat} \underrightarrow{T}$ are represented by the given ones in $\operatorname{Cat} T_j$ and the functor $j\backslash J \longrightarrow (\text{Sets})$ is defined as before (although $j\backslash J \neq j, j\backslash J$).

The proof is routine.

Hence $\underrightarrow{T}$ is a topology, and it is clear that the functors $L_i : \operatorname{Cat} T_i \longrightarrow \operatorname{Cat} \underrightarrow{T}$ are morphisms of topologies and that $(\underrightarrow{T}, L)$ represents $\varinjlim T$.

Let $T : J \longrightarrow (\text{Top})$ be given and fix $0 \in J$, F a presheaf on T_0. For $0 \xrightarrow{g} j$ in J we get a presheaf $(g_t)_p F$ on T_j, and using the functor $L_0 : T_0 \longrightarrow \underrightarrow{T}$ we get a presheaf $\underrightarrow{F} = (L_0)_p F$ on $\underrightarrow{T}$. We have, for $U \in \operatorname{Cat} \underrightarrow{T}$, $\underrightarrow{F}(U) = \varinjlim_{(X, \emptyset) \in I_U^{L_0}} F(X)$ where $I_U^{L_0}$ is the category of pairs $(X, \emptyset)$, $X \in \operatorname{Cat} T_0$, $\emptyset \in \operatorname{Hom}_{\underrightarrow{T}}(U, X)$ (cf. I (2.2)). Say U was in $\operatorname{Cat} T_i$. Then for

$$\begin{array}{ccc} 0 & \overset{g}{\searrow} & \\ & & j \\ i & \underset{g'}{\nearrow} & \end{array} \quad \in 0, i\backslash J$$

we have

$$[(g_t)_p \mathcal{F}]\,(g'_t\,(U)) \;=\; \varinjlim_{(X,\phi_j)} \mathcal{F}(X)$$

where the limit is over $(X,\phi_j) \in I^{g_t}_{g'_t(U)}$ (the category of pairs (X, ϕ_j), $X \in \mathrm{Cat}\, T_0$, $\phi_j \in \mathrm{Hom}\,(g'_t(U),\, g_t(X))$. Suppose $j \xrightarrow{f} k$ in J. Define a functor

$$(3.4) \qquad \bar{f} \;:\; I^{g_t}_{g'_t(U)} \longrightarrow I^{fg_t}_{fg'_t(U)} \qquad \text{by } (X,\phi_j) \rightsquigarrow (X,\phi_k)$$

where

$$\phi_k = t_{f,g}(X) \circ f_t(\phi_j) \circ t^{-1}_{f,g'}(U)$$

and by the identity map on morphisms. One obtains in this way a <u>functor</u> $I_U : 0,\, i \backslash J \longrightarrow (\mathrm{Cat})$ carrying

$$\left(0 \xrightarrow{\;g\;} j \xleftarrow{\;g'\;} i\right) \rightsquigarrow I^{g_t}_{g'_t(U)}$$

and it is easily seen that $\varinjlim I_U \xrightarrow{\sim} I_U^{L_0}$ (where the $\varinjlim$ is in the sense of honest functors, not pseudofunctors).

The case in which we are interested is the following: Fix a topology T_0 with finite covering families, and a category J satisfying (L1, 2, 3). Let $V : J^0 \longrightarrow \mathrm{Cat}\, T$ be a functor $(i \rightsquigarrow V_i)$. We suppose that for $i, j \in J$ products of the form $V_i \underset{V_j}{\times} U$, $V_i \times U$ exist for $U \in \mathrm{Cat}\, T_0$, and hence obtain a pseudofunctor $T : J \longrightarrow (\mathrm{Top})$ by $i \rightsquigarrow T/V_i$ (cf. II Definition 4.12), and for $i \xrightarrow{f} j$ letting $f_t : T_i \longrightarrow T_j$ be the functor $U/V_i \rightsquigarrow U \underset{V_i}{\times} V_j/V_j$. The f_t's are morphisms of topologies and the isomorphisms $t_{f,g}$ are given by formula (3.3.9.1) of EGAI for fibred products. To avoid overloading notation we denote also by J the category obtained from J by adjoining an initial element 0, and we extend T by sending $0 \rightsquigarrow T_0$ with $T_0 \xrightarrow{g_t} T_i$ defined by $U \rightsquigarrow U \times V_i$. The topologies T_j all have finite covering families and so $\underset{\rightsquigarrow}{\varinjlim}\, T = \underset{\rightarrow}{T}$ exists.

Let as above $L_0 : T_0 \longrightarrow \underset{\rightarrow}{T}$ be the functor and for a presheaf on T_0 write $\underset{\rightarrow}{F} = (L_0)_p F$, $F_i = (g_t)_p F$ for $0 \xrightarrow{g} i$. Notice that $0, i\backslash J$ is canonically isomorphic to $i\backslash J$.

THEOREM (3.5): Make the above assumptions and notations:

(i) Let $U \in \mathrm{Cat}\, \underset{\rightarrow}{T}$, say U is in Cat T_i. Then for anpresheaf F on T_0,

$$\underset{\rightarrow}{F}(U) \approx \varinjlim_{i \xrightarrow{g'} j \in i\backslash J} F_i(g_t'(U)) \approx \varinjlim_{i \longrightarrow j \in i\backslash J} F(U \underset{V_i}{\times} V_j) .$$

(ii) If F is a sheaf then $\underset{\rightarrow}{F}$ is a sheaf, and if F is flash so is $\underset{\rightarrow}{F}$.

(iii) The functors $(L_0)_p$ and $(L_0)_s \approx (L_0)_p \,|\, \mathcal{S}_0$ are exact ($\mathcal{S}_0$ = sheaves on T_0).

(iv) If $\mathbb{F}$ is a sheaf and $U \in \mathrm{Cat}\, T_0$ then

$$H^p(\underset{\rightarrow}{T}, U; \underset{\rightarrow}{F}) \approx \varinjlim_{j \in J} H^p(T_j, U \times V_j, F_j) \approx \varinjlim_{j \in J} H^p(T_0, U \times V_j ; F) .$$

<u>Proof:</u> We have first to explain the categories over which the limits are taken. To begin with, note that for $i \xrightarrow{f} j$ the functors $(f_t)_p$ has a very special form due to the fact that f_t has as left adjoint the functor $\epsilon : U/V_j \rightsquigarrow U/V_i$ (by composition with $U_j \longrightarrow U_i$). In fact, by I (2.8) we have canonically $(f_t)_p \approx \epsilon^p$. Thus if the functor $i \xrightarrow{g'} j \rightsquigarrow F(U \underset{V_i}{\times} V_j)$ is defined in the obvious way we obtain also a functor $i \xrightarrow{g'} j \rightsquigarrow F_i(g_t'(U))$ and so an isomorphism of the limits of (i). This second functor is the same as the one induced by (3.4). The second isomorphism of (iv) is clear from II (4.13) if the functors are defined using those of (i) by universality of H^p(nb. $0 \backslash J = J$) and the first isomorphism of (iv) will be trivial if (i), (ii), (iii) are known.

Revert to the notation of (3.4). The categories $I^{g_t}_{g_t'(U)}$ satisfy $(L1, 2, 3)^0$, indeed, they have initial objects (cf. I (2.8)), and it follows readily that $I^{L_0}_U$ also satisfies $(L1, 2, 3)^0$. This proves (iii) (cf. Proof of II Theorem 4.14). The functors $(I^{g_t}_{g_t'(U)})^0 \longrightarrow (Ab)$, $((X, \phi_j) \rightsquigarrow F(X))$, are clearly obtained by composition with the functor $I^{L_0}_U \longrightarrow (Ab)$, $((X, \phi) \rightsquigarrow F(X))$. So to complete the proof of (i) one has only to check

LEMMA (3.6): Let $C : B \longrightarrow (\mathrm{Cat})$ be a functor, suppose B and the C_i satisfy (L1, 2, 3) and set $\underset{\rightarrow}{C} = \varinjlim C$. Let $F : \underset{\rightarrow}{C} \longrightarrow D$ be a functor and define $F_i : C_i \longrightarrow D$ by composition. Then

$$\varinjlim_{B} \left(\varinjlim_{C_i} F_i \right) \xrightarrow{\sim} \varinjlim_{\underset{\rightarrow}{C}} F \quad .$$

To prove (ii), let $\mathcal{F}$ be a sheaf, and $\{U_\alpha \longrightarrow U\} \in \mathrm{Cov}\ \underset{\rightarrow}{T}$. There are finitely many α, and so we may replace the objects and maps by an isomorphic situation so that the U_α, U are in Cat T_0 and the maps are represented in T_0 with $\{U_\alpha \longrightarrow U\} \in \mathrm{Cov}\ T_0$. Then we have for all V_j, $\{U_\alpha \times V_j \longrightarrow V_j\} \in \mathrm{Cov}\ T_0$. Hence since $\mathcal{F}$ is a sheaf

$$\begin{array}{l} \mathcal{F}(U \times V_j) \longrightarrow \prod_{\alpha} \mathcal{F}(U_\alpha \times V_j) \rightrightarrows \prod \mathcal{F}(U_\alpha \times U_\beta \times V_j) \\ = \\ \mathcal{F}(U \times V_j) \longrightarrow \bigoplus \mathcal{F}(U_\alpha \times V_j) \rightrightarrows \bigoplus \mathcal{F}(U_\alpha \times U_\beta \times V_j) \end{array}$$

is exact. Since $\varinjlim$ commutes with $\oplus$ and is exact for (L1, 2, 3) it follows from (i) that $\underset{\rightarrow}{\mathcal{F}}$ is a sheaf. The flaskness is obtained in the same way. (One can also show $\underset{\rightarrow}{\mathcal{F}}$ is a sheaf of sets if $\mathcal{F}$ is by using I (1.8) instead of $\prod = \oplus$).

Now let $X : J^0 \longrightarrow (\text{preschemes})$ be a functor $(j \rightsquigarrow X_j)$. Assume still J satisfies (L1, 2, 3). Denote by $S_{X_j} = S_j$ the category of preschemes over X_j separated and of finite presentation. Given

$i \longrightarrow j$, so that $X_i \longleftarrow X_j$, one gets a functor $S_i \longrightarrow S_j$ by base extension, and hence a pseudofunctor $S : J \longrightarrow \mathrm{Cat}$ $(j \rightsquigarrow S_j)$. If for all $i \longrightarrow j$ in J the maps $X_i \longleftarrow X_j$ are affine, one can show that $\varprojlim X$ is representable in the category of preschemes, and affine over the X_j's. Let $\underleftarrow{X} = \varprojlim X$ and call $S_{\underleftarrow{X}}$ the category of preschemes over $\underleftarrow{X}$ separated, of finite presentation. Base extension to $\underleftarrow{X}$ yields a morphism $S \longrightarrow \mathrm{const}_{S_{\underleftarrow{X}}}$ of pseudofunctors, hence a functor $\varinjlim S \longrightarrow S_{\underleftarrow{X}}$.

THEOREM (3. 7): Let $X : J^0 \longrightarrow$ (preschemes) be a functor, J satisfying (L1, 2, 3). Suppose that for $j \in J$, X_j is quasicompact and quasiseparated, and that for $i \longrightarrow j$ in J the maps $X_i \longleftarrow X_j$ are affine. Let $\underleftarrow{X} = \varprojlim X$. Then with the above notation the functor $\varinjlim S \longrightarrow S_{\underleftarrow{X}}$ is an equivalence of categories.

The proof will be in EGA IV (if J is inductive, which is enough for the applications).

Let T_j be the case (f) topology on X_j, $T_{\underleftarrow{X}}$ the case (f) topology on $\underleftarrow{X}$. One obtains in the same way a pseudofunctor $T : J \longrightarrow (\mathrm{Top})$, $(j \rightsquigarrow T_j)$ and a morphism of topologies $\varinjlim T \longrightarrow T_{\underleftarrow{X}}$.

THEOREM (3. 8): Make the same assumptions as in (3. 7). Then $\varinjlim T \longrightarrow T_{\underleftarrow{X}}$ is an equivalence of topologies.

... en by an equivalence of topologies $F : T_1 \longrightarrow T_2$ ... morphism which is an equ... categories $\mathrm{Cat}\ T_1 \longrightarrow \mathrm{Cat}\ T_2$ and such that a family $\{U_\alpha \longrightarrow U\}$ of $\mathrm{Cat}\ T_1$ is in $\mathrm{Cov}\ T_1$ if and only if $\{F(U_\alpha) \longrightarrow F(U)\}$ is in $\mathrm{Cov}\ T_2$. It is clear that an equivalence induces equivalences on the categories of presheaves and sheaves.

Proof of Theorem (3.8): $\mathrm{Cat}\ T_i$ is a full subcategory of S_i and $\mathrm{Cat}\ T_{\underleftarrow{X}}$ is a full subtopology of $S_{\underleftarrow{X}}$. Hence $\mathrm{Cat} \varinjlim T$ is a full subcategory of $\varinjlim S$. Since obviously the morphism $\varinjlim T \longrightarrow T_{\underleftarrow{X}}$ is induced by the functor $\varinjlim S \longrightarrow S_{\underleftarrow{X}}$ which is an equivalence, we know anyhow that $\mathrm{Cat} \varinjlim T \longrightarrow \mathrm{Cat}\ T_{\underleftarrow{X}}$ is fully faithful. We have to check

(a) Given $U \in \mathrm{Cat}\ T_{\underleftarrow{X}}$, $\exists\, j \in J$, $U_j \in \mathrm{Cat}\ T_j$ and an isom. $U_j \times_{X_j} \underleftarrow{X} \approx U$.

(b) If $\{U \longrightarrow V\} \in \mathrm{Cov}\ T_{\underleftarrow{X}}$ then $\exists\, j \in J$, $\{U_j \longrightarrow V_j\} \in \mathrm{Cov}\ T_j$ and isomorphisms $U_j \times_{X_j} \underleftarrow{X} \approx U$, $V_j \times_{X_j} \underleftarrow{X} \approx V$ which commute with the induced maps. (Since the covering families are finite we have replaced $\{U_\alpha \longrightarrow V\}$ by the single map $\coprod U_\alpha = U \longrightarrow V$.)

To show (a), one reduces readily to the case $X_i = \mathrm{spec}\ A_i$, $\underleftarrow{X} = \mathrm{spec}\ \underrightarrow{A}$, $U = \mathrm{spec}\ B$ affine. Then choose B_i/A_i finitely presented, some i, so that $B_i \otimes_{A_i} \underrightarrow{A} \approx B$, and set $B_j = B_i \otimes_{A_i} A_j$ for $i \longrightarrow j$. A theorem of Grothendieck guarantees that B_j/A_j is flat for some j.

We need B_j/A_j étale, i.e., we have to show $\Omega^1_{B_j/A_j} = \Omega^1_j = 0$ for some j (and we know $\Omega^1_{B/\underrightarrow{A}} = 0$). But since B_j/A_j is of finite type, so is Ω^1_j (as B_j - module). We have $\Omega^1_{B/\underrightarrow{A}} \approx \Omega^1_i \otimes_{A_i} \underrightarrow{A}$ $\approx \varinjlim \Omega^1_i \otimes_{A_i} A_j$. If $\{\omega_\nu\}$ are a set of generators for Ω^1_i, so that $\{\omega_\nu \otimes 1_j\}$ generate Ω^1_j it is clear that these elements must be zero for some j, as desired.

To show (b), let $U \longrightarrow V$ be surjective in Cat $T_{\underleftarrow{X}}$, and choose i, $U_i \longrightarrow V_i$ in Cat T_i with $(U_i \longrightarrow V_i) \times_{X_i} \underleftarrow{X} \approx U \longrightarrow V$ using (a). Since $U_i \longrightarrow V_i$ is étale, finitely presented, the image $\widetilde{V}_i$ of U_i is open in V_i. One sees easily that $\widetilde{V}_i \longrightarrow V_i$ is also finitely presented, so $U_i \longrightarrow \widetilde{V}_i \in \text{Cov } T_i$. Let $\widetilde{V} = V \times_{X_i} \underleftarrow{X}$. Clearly $\widetilde{V} \approx V$, hence replacing V_i by $\widetilde{V}_i$ we are done.

COROLLARY (3.9): Let X_0 be a prescheme, $X : J^0 \longrightarrow \text{Cat } T_{X_0}$ ($j \rightsquigarrow X_j$) a functor, where the topology is any of the cases (f, 0, 1, 2). Suppose the X_j quasi compact and quasi separated and that the maps $X_i \longleftarrow X_j$ are affine. Suppose finally J satisfies (L1, 2, 3). Let $\underleftarrow{X} = \varprojlim X$ and for a sheaf F on X_0, let F_i, $\underrightarrow{F}$ be the sheaves induced on X_i resp. $\underleftarrow{X}$. Then

$$\varinjlim_J H^q(X_i, F_i) \xrightarrow{\sim} H^q(\underleftarrow{X}, \underrightarrow{F}) .$$

Clear from (1.1), (3.5), and (3.8). Note that if $\pi : \underleftarrow{X} \longrightarrow X$ is the map, $\underrightarrow{F} = \pi^* F$.

Section 4. <u>Hensel rings.</u> We include for convenience some basic facts on Hensel rings. Most of the results and proofs are taken from papers of Azumaya (A) and Nagata (N).

Let A be a local ring, $\mathfrak{m}$ its maximal ideal. Consider the category $\Xi(A)$ of rings B local over A, B obtained by localizing an A-algebra of finite presentation with B/A étale and $B/\mathfrak{m}B \approx A/\mathfrak{m}A$. This category is an inductive system (cf. SGA I, Corollary 5.4). Set

$$\widetilde{A} = \varinjlim_{B \in \Xi(A)} B .$$

$\widetilde{A}$ is naturally a functor of A. Clearly $\widetilde{A}$ is local, $\widetilde{A}/A$ is étale, and $\widetilde{A}/\mathfrak{m}\widetilde{A} \approx A/\mathfrak{m}A$. $\widetilde{A}$ has the following property:

(4.1) If $B/\widetilde{A}$ is local, étale, localized from an $\widetilde{A}$-algebra of finite presentation, and if $B/\widetilde{\mathfrak{m}}B \approx \widetilde{A}/\widetilde{\mathfrak{m}}\widetilde{A}$ then $\widetilde{A} \approx B$.

Or, equivalently,

(4.1') If $f \in \widetilde{A}[t]$ is monic, and if $\bar{a} \in \widetilde{A}/\widetilde{\mathfrak{m}}\widetilde{A}$ is a simple root of the polynomial $\bar{f} \in \widetilde{A}/\widetilde{\mathfrak{m}}\widetilde{A}[t]$ (the image of f), then f has a simple root a inducing $\bar{a}$.

To see that (4.1) $\Longrightarrow$ (4.1'), set $B = (\widetilde{A}[t]/f)_{\mathfrak{q}}$ where $\mathfrak{q}$ is the maximal ideal induced by the root $\bar{a}$ of $\bar{f}$. We get $B \approx \widetilde{A}$. Hence the image of t in B yields the desired root in $\widetilde{A}$.

For the converse, use SGAI, Theorem 7.6 to write $B \approx (\tilde{A}[t]/f)_{\mathfrak{q}}$ where f is monic and $\mathfrak{q}$ is a maximal ideal over $\tilde{\mathfrak{m}}$ (the theorem extends to the non-noetherian case, if finite presentation is substituted for finite type). Obviously $B/\tilde{\mathfrak{m}}\, B \approx \tilde{A}/\tilde{\mathfrak{m}}\, \tilde{A}$ means $\bar{f}$ has a simple root. Hence f has a simple root inducing it and one finds $\tilde{A}[t]/f \approx \tilde{A} \times C$ for some C, so $B \approx \tilde{A}$. It is now clear that $\tilde{A}$ has property (4.1').

THEOREM (4.2): If A is noetherian, so is $\tilde{A}$.

This elegant proof is due to Nagata (N): Clearly $A/\mathfrak{m}^n A \approx \tilde{A}/\tilde{\mathfrak{m}}^n \tilde{A}$, hence A and $\tilde{A}$ have the same completion $\hat{A}$. Since $\tilde{A}$ is a limit of noetherian local rings (which are separated for the $\mathfrak{m}$-adic topology), A is separated, so we have $A \subset \tilde{A} \subset \hat{A}$. One uses the following criterion for noetherian rings: every ascending chain $\cdots \subseteq \mathfrak{a}_i \subseteq \cdots$ of finitely generated ideals becomes constant. Since $\hat{A}$ is noetherian the chain $\cdots \subseteq \mathfrak{a}_i \hat{A} \subseteq \cdots$ of ideals of $\hat{A}$ becomes constant, hence the chain $\cdots \subset \mathfrak{a}_i \hat{A} \cap \tilde{A} \subseteq \cdots$ becomes constant. So it suffices to show $\mathfrak{a}\hat{A} \cap \tilde{A} = \mathfrak{a}$ for finitely generated ideals $\mathfrak{a}$, and it is obvious that $\mathfrak{a}\hat{A} \cap A \supseteq \mathfrak{a}$. Let $\{a_1, \cdots, a_n\}$ generate $\mathfrak{a}$, and let $b \in \mathfrak{a}\hat{A} \cap \tilde{A}$. Choose $B/A \in E(A)$ so that $a_1, \cdots, a_n,\ b \in B$ and set $\mathfrak{a}_0 = (a_1, \cdots, a_n)B$. Since B is noetherian with completion $\hat{A}$, it is well known that $\mathfrak{a}_0 \hat{A} \cap B = \mathfrak{a}_0$. But $\mathfrak{a}_0 \hat{A} \cap B = \mathfrak{a}\hat{A} \cap B$ contains b. Hence $b \in \mathfrak{a}_0$ and so $b \in \mathfrak{a}$.

THEOREM (4.3): Let A be a local ring, $\mathfrak{m}$ its maximal ideal, $\bar{A} = A/\mathfrak{m}A$. The following are equivalent:

(i) If $f \in A[t]$ is a monic polynomial such that its image $\bar{f}$ in $\bar{A}[t]$ has the form $\bar{f} = \bar{g}\bar{h}$ with $\bar{g}, \bar{h}$ monic and $(\bar{g}, \bar{h}) = 1$ there exist unique monic polynomials $g, h \in A[t]$ representing $\bar{g}, \bar{h}$ respectively such that $f = gh$, $(g,h) = 1$.

(ii) Every finite A-algebra is a direct product of local rings.

(iii) A $(=\tilde{A})$ has property (4.1).

(iv) Definition: A is a Hensel ring.

Proof: (i) $\Longrightarrow$ (ii): Let B/A, finite, be given, and decompose $\bar{B} = B/\mathfrak{m}B$ (which is finite over the field $\bar{A}$) into a direct product of local rings. We have to find a corresponding decomposition of B into a direct product. So if $\bar{e}$ is an idempotent of $\bar{B}$, we have to find an idempotent of B lying over $\bar{e}$. Let $b \in B$ be any element lying over $\bar{e}$. Clearly we may assume b generates B. Then $\bar{B} \approx \bar{A}[t]/\bar{f}$ for some monic polynomial $\bar{f}$, hence by Nakayama B is a quotient of $A[t]/f$ with f monic representing $\bar{f}$. But the decomposition of $\bar{B}$ into a direct product corresponds to the factorization of $\bar{f}$ into relatively prime factors. Since (i) holds, f factors correspondingly, so we are done.

(ii) $\Longrightarrow$ (iii): If B/A is given so as to test (4.1), write $B \approx C_{\mathfrak{q}}$ where C/A is finite. Applying (ii), $C \approx B \times D$ for some D. Hence B is finite (and étale). Since $A/\mathfrak{m}A \approx B/\mathfrak{m}B$, B = A by Nakayama.

(iii) $\Longrightarrow$ (i): Suppose A satisfies (4.1). Write $\mathbb{Z}[|A|] \longrightarrow A \longrightarrow 0$ where $|A|$ denotes the set of elements of A, and let $\mathfrak{p} \subset \mathbb{Z}[|A|]$ be the inverse image of $\mathfrak{m}$. Set $A' = \mathbb{Z}[|A|]_{\mathfrak{p}}$, so that $A' \longrightarrow A$ is surjective and local. Since $A \approx \widetilde{A}$, we get $\widetilde{A}' \longrightarrow A$. Now it is clear that (i) is preserved in quotients, so we are reduced to showing (i) for $\widetilde{A}'$. Let $f \in \widetilde{A}'[t]$ be given, monic, with $\bar{f} = \bar{g}\bar{h}$, $(\bar{g}, \bar{h}) = 1$ in $\bar{A}'[t]$. f is represented in $B'[t]$ for some B'/A', $B' \in \mathcal{E}(A')$. Applying SGAI, Theorem 7.6, we may write $B' \approx C'_{\mathfrak{q}}$ where $C' = A'[t]/q(t)$. Choose finitely many elements $a_1, \cdots, a_n$ of $|A|$ so that, if $A_0 = (\mathbb{Z}[a_1, \cdots, a_n])_{\mathfrak{p}_0}$ where $\mathfrak{p}_0$ is the prime ideal induced by $\mathfrak{p}$, the coefficients of $q(t)$ are in A_0, $\bar{q}(t)$ has the simple root $\bar{a}$ in $\bar{A}_0$ inducing $\mathfrak{q}$, and $\bar{f}(t)$ factors as in $\bar{A}'[t]$. Set $B_0 = (A_0[t]/q(t))_{\mathfrak{q}_0}$ ($\mathfrak{q}_0$ induced by $(t - \bar{a})$). Then $B_0 \subset \widetilde{A}_0$. We may finally assume also that f is represented in $B_0[t]$, hence represented in $\widetilde{A}_0[t]$. Since $\widetilde{A}_0 \longrightarrow \widetilde{A}'$ we are reduced to proving (i) for $\widetilde{A}_0$, where A_0 is noetherian and regular. Then $\widetilde{A}_0$ is also noetherian by (4.2) and regular (since it has a regular completion). So we assume A given satisfying (4.1), and A noetherian and regular. Let f be monic in $A[t]$. We may assume f irreducible and separable over the field K of fractions of A, and we have to show $A[t]/f$ is a local ring. Let L/K be a Galois extension in which f splits, and B the normalization of A in L. Write $A[t]/f \subset B$. It suffices to show B is local. Let $\mathfrak{P}$ be a prime of B above $\mathfrak{p}$, and let B_0 be the decomposition ring of $\mathfrak{P}$ with respect to the operation of $G(L/K)$, $\mathfrak{p}_0$ the maximal ideal of B_0

under $\mathfrak{p}$. Then $B_{0\mathfrak{p}_0}$ is étale /A, with no residue field extension (SGA V, Proposition 2. 2). Hence $B_{0\mathfrak{p}_0} = A$, hence $B_0 = A$. But this means $\mathfrak{p}$ is the only prime of B over $\mathfrak{m}$.

COROLLARY (4. 4): (i) If A is a hensel ring and $\mathfrak{a}$ is an ideal of A then $A/\mathfrak{a}$ is a hensel ring.

(ii) If A is a hensel ring and B/A is finite and local then B is a hensel ring.

(iii) If A is any local ring then $\tilde{A}$ is a hensel ring.

For (i), (ii), (iii) apply (i), (ii), (iii) respectively of (4. 3).

<u>Terminology</u> (4. 5): $\tilde{A}$ is the <u>henselization</u> of A.

The map $A \longrightarrow \tilde{A}$ is universal with respect to maps into hensel rings. As a functor of A, $\tilde{A}$ has very good properties, better in fact than the completion. For instance, supposing A noetherian, we have

A regular iff. $\tilde{A}$ regular

A reduced iff. $\tilde{A}$ reduced

A normal iff $\tilde{A}$ normal.

These properties, and others, are easily deduced from SGA I, ection 9.

THEOREM (4.6): Let A be a hensel ring. The category of finite and finitely presented, étale A-algebras is equivalent with the category of finite separable $\bar{A}$-algebras. ($\bar{A} = A/\mathfrak{m}A$).

Proof: The equivalence is of course given by $B \rightsquigarrow \bar{B} = B/\mathfrak{m}B$. To show that every separable $\bar{A}$-algebra L is isomorphis to a suitable $\bar{B}$, we may assume L is a separable field extension, so $L \approx \bar{A}[t]/\bar{f}$ for some monic $\bar{f}$. Let $f \in A[t]$ (monic) represent $\bar{f}$, and set $B = A[t]/f$, so that $\bar{B} \approx L$. Since $(\bar{f}') = 1$ in L, we have $(f') = 1$ in B by Nakayama. Hence B/A is étale, finite, finitely presented, as desired.

We have to show that the functor is bijective on Hom. Let B, C be given /A. We may assume Spec C connected, and it follows from SGA I, Corollary 5.4 that $\mathrm{Hom}_A(B,C) \subset \mathrm{Hom}_{\bar{A}}(\bar{B},\bar{C})$. Suppose given a map $\bar{B} \longrightarrow \bar{C}$ and consider the graph $\bar{B} \otimes_{\bar{A}} \bar{C} \longrightarrow \bar{C}$. Since $\bar{B}, \bar{C}$ are separable over $\bar{A}$, $\bar{B} \otimes_{\bar{A}} \bar{C} \approx \bar{C} \times L$ for some L in such a way that the graph is the projection (Spec $\bar{C}$ connected by (4.3) (ii)). Since $B \otimes_A C$ is finite over A, it follows that $B \otimes_A C \approx C' \times D$ for some C', D with $\bar{C}' \approx \bar{C}$, $\bar{D} \approx L$. Combining the projection with the map $C \longrightarrow B \otimes_A C$ we get a map $C \longrightarrow C'$, and the induced map $\bar{C} \longrightarrow \bar{C}'$ is an isomorphism. We have to show $C \longrightarrow C'$ is an isomorphism. But C' is finite, finitely presented, and flat, hence free. So a set of elements of C' which is a basis mod $\mathfrak{m}$ of $\bar{C}'$ as an $\bar{A}$-module is a basis for C' as an A-module. Therefore the images in C' of a basis for C is a basis for C', so $C \xrightarrow{\sim} C'$.

Let G be a profinite group and denote by T_G the topology of finite continuous G-sets (cf. I(0.6) bis). Let X be a prescheme and $f : T_G \longrightarrow T_X$ a morphism. Since the category of T_G-sheaves is equivalent with the category of continuous G-modules, we may identify $f^s F$, $F \in \mathcal{S}_X$, with a certain G-module. The underlying group of this G-module is easily seen to be $\varinjlim_{\overline{G}} F(f(\overline{G}))$ where $\overline{G}$ runs over the category of quotients of G with respect to open normal subgroups. Since the functor associating with a G-module its underlying group is exact, it is clear that the underlying group of the module corresponding to $R^q f^s F$ is

$$R^q \varinjlim_{\overline{G}} F(f(\overline{G})) = \varinjlim_{\overline{G}} H^q(T_X, f(\overline{G}) ; F) .$$

So II(4.11) reads (where $e \in \mathrm{Cat}\ T_G$ is the set of one element)

HOCHSCHILD - SERRE SPECTRAL SEQUENCE (4.7):

$$E_2^{p,q} = H^p\left(G, \varinjlim_{\overline{G}} H^q(T_X , f(\overline{G}); F)\right) \Longrightarrow H^*(T_X, f(e); F) .$$

Here the G-cohomology is the Tate cohomology. For instance, G might be $\pi_1(X)$, or a quotient by a closed normal subgroup of $\pi_1(X)$. In such cases one can often apply Corollary 3.9 to interpret the limit. Thus if X is quasicompact and quasiseparated, X given over a field k, and if $\overline{X} = X \otimes_k \overline{k}$ ($\overline{k}$ the separable algebraic closure of k), and $\pi : \overline{X} \longrightarrow X$ is the map, one gets a spectral sequence

$$(4.8) \qquad H^p(G(\bar{k}/k), H^q(\bar{X}, \pi^* F)) \Longrightarrow H^*(X; F) \quad .$$

Let again A be a hensel ring and set $k = A/\mathfrak{m} A$, $G = G(\bar{k}/k)$. Theorem (4.6) and Galois theory yield a morphism $f : T_G \longrightarrow T_X$ where $X = \operatorname{Spec} A$ and the topology in case (f).

THEOREM (4.9): Let A be a hensel ring, $X = \operatorname{Spec} A$, $k = A/\mathfrak{m} A$, $F \in \mathcal{S}_X$. With the above notation

$$H^p(G(\bar{k}/k), f^s F) \approx H^p(X, F) \quad .$$

In particular, if k is separably algebraically closed, then $H^p(X;F) = 0$ for all $F \in \mathcal{S}_X$, $p > 0$.

Proof: By (4.7), we have to show

$$R^q f^s F = \varinjlim_{\bar{G}} H^q(T_X, f(\bar{G}); F) = 0, \qquad q > 0 .$$

Set $\underset{\leftarrow}{X} = \varprojlim_{\bar{G}} f(\bar{G})$. $\underset{\leftarrow}{X}$ is affine with ring $\underset{\rightarrow}{A} = \varinjlim A(\bar{G})$, $A(\bar{G})$ the affine ring of $f(\bar{G})$. Each ring $A(\bar{G})$ corresponds to a finite Galois extension $\bar{k} \supset k(\bar{G}) \supset k$ and hence is connected, therefore local by (4.3)(ii) and hensel by (4.4)(ii). It is clear that therefore $\underset{\rightarrow}{A}$ is local and hensel, with residue field $\bar{k}$. Let $\underset{\rightarrow}{F}$ be the sheaf induced on $\underset{\leftarrow}{X}$ by F. By Corollary (3.9), $R^q f^s F \approx H^q(\underset{\leftarrow}{X}; \underset{\rightarrow}{F})$. Hence we are reduced to the case $k = \bar{k}$ separably algebraically closed.

Let U/X be étale, finitely presented, and suppose U covers the closed point of X (so that U covers X). Let B be the local ring of U at a point above the closed point. Then by SGA I, Theorem 7.6 and (4.3)(ii), B is étale, finite, and fin. pres.. Since $k = \bar{k}$, $B \approx A$ by (4.6). Therefore X is initial in the category of $U \in \text{Cat } T_X$ which cover X. To show $H^q(X, F) = 0$, consider the identity map $id : X \longrightarrow X$. Clearly $R^q id F = 0$, $q > 0$. Hence the stalk of $R^q id_* F$ is zero at every geometric point. Let $P \longrightarrow X$ be a geometric point of X above the closed point and apply II Corollary (4.7) and III Proposition (1.9). Since X is initial in the category of U covering X, we find $(R^q id_* F)_p \approx H^q(X, F)$, therefore $H^q(X, F) = 0$, $q > 0$.

LEMMA (4.10): Let X be a prescheme and $\epsilon : P \longrightarrow X$ a geometric point. Let I be the category of pairs $(U, \emptyset)$; $U \in \text{Cat } T_X$, $\emptyset : P \longrightarrow U$ over ϵ in the case (1) topology. The subcategory I' of I of pairs $(U, \emptyset)$ with U affine is initial in I, and $\varprojlim_{(U,\emptyset) \in I'} U$ is the spectrum of a hensel ring with separably algebraically closed residue field (the residue field is the separable algebraic closure of $k(x)$ if $x \in X$ is the center of P).

Routine .

COROLLARY (4.11): Let $\pi : Y \longrightarrow X$ be a finite morphism. Assume case (1). Then $R^q \pi_* F = 0$ for all $F \in \mathcal{S}_Y$, $q > 0$.

<u>Proof:</u> We have to show the stalks are zero. Let $\epsilon : P \longrightarrow X$ be a geometric point. By II(4.7) and Proposition (1.9),

$$(R^q\pi_*F)_P \approx \varinjlim_{(U,\phi)\in I} H^q(T_Y, U \underset{X}{\times} Y; F)$$

$$\approx \varinjlim_{(U,\phi)\in I'} H_q^q(T_Y, U \underset{X}{\times} Y; F) \quad \text{where } I, I'$$

are as in (4.10). Since π is finite, it is affine, and therefore $U \underset{X}{\times} Y$ is affine for all $(U, \phi) \in I'$. Set $\operatorname{Spec} A = \varprojlim_{I'} U$; $\operatorname{Spec} B = \varprojlim_{I'} U \underset{X}{\times} Y$. We have $(R^q\pi_*F)_P \approx H^q(\operatorname{Spec} B, \underrightarrow{F})$ by Corollary (3.9) for suitable $\underrightarrow{F}$. B/A is finite, and so B is a direct product of hensel rings by (4.10), (4.3)(ii), (4.4)(ii), and the hensel rings have separably algebraically closed residue fields. Since cohomology obviously commutes with direct sums of schemes, $H^q(\operatorname{Spec} B, \underrightarrow{F}) = 0$, $q > 0$ by (4.9), so we are done.

Section 5. <u>Cohomological dimension.</u> Let T be a topology, and F a presheaf of abelian groups on T. F is called a <u>torsion</u> presheaf if and only if F(U) is a torsion group for all $U \in \operatorname{Cat} T$.

PROPOSITION (5.1): Let $f : T' \longrightarrow T$ be a morphism of topologies and F a torsion sheaf on T. Suppose T is noetherian (cf. II, Section 5). Then R^qf^sF is a torsion sheaf on T' for all $q \geq 0$.

Proof: Clearly, an inductive limit of torsion groups is a torsion group. Recalling (II Section 1) the construction of the functor $\#$, we are reduced to showing $\mathcal{R}^q F$ is a torsion presheaf where $\mathcal{R}^q F$ is as in II Corollary (4.7), i.e., we are reduced to showing $H^q(T, U; F)$ is a torsion group for all $q \geq 0$, $U \in \mathrm{Cat}\ T$. Write

$$F = \sup_{n \in \mathbb{N}} F_n$$

where F_n is the kernel of multiplication by n in F (so that F_n is the subsheaf of F of sections whose order divides n). By II (5.4) we are reduced to the case $F = F_n$, since T is noetherian. But the multiplication $F \xrightarrow{n} F$ induces multiplication $H^q(T, U; F) \xrightarrow{n} H^q(T, U; F)$, as is seen by multiplying an injective resolution of F by n. Since for $F = F_n$ the multiplication is the zero map, it follows that $H^q(T, U; F)$ is annihilated by n.

COROLLARY (5.2): Let $\pi : X \longrightarrow Y$ be a morphism of preschemes, X and Y quasi compact and quasiseparated, and let F be a torsion sheaf on X. Then $R^q \pi_* F$ is a torsion sheaf on Y, all $q \geq 0$.

This is trivial in case (f) because then T_X is noetherian. For the other cases, one has to verify (notation of Section 1) that a sheaf is torsion if and only if $\alpha^s F$(resp. $\beta^s F$, resp. $\gamma^s F$) is torsion, under the assumption (by (1.1)) that these functors are equivalences of categories. The "only if" is obvious. To show " if " we need, since $\alpha_s \alpha^s F \xrightarrow{\sim} F$ (resp. ...) to show that if G is a torsion sheaf then $\alpha_s G$ is a torsion sheaf. But the construction of $\alpha_s G$ (cf. II Section 4, I (2.1), II (1.1)) is by means of inductive limits and so it is clear.

Let X be a prescheme, F a sheaf on X, and $i : x \longrightarrow X$ a point of X, where x = Spec k(x) is given the structure of scheme. F is said to be <u>zero at</u> x iff. i^*F is the zero sheaf. Obviously, if $\epsilon : P \longrightarrow X$ is a geometric point centered at x then the stalk (cf. (1.7)) of F at P is isomorphic to the stalk of i^*F at "P". One sees easily that F is zero at x iff. the stalk of F at P is zero for any geometric point $\epsilon : P \longrightarrow X$ centered at x.

Suppose X defined over a field k. For a point x of X, write dim x = tr. $\deg._k k(x)$ (so dim x is the reference to k). We say a sheaf F on X is <u>zero in dimension</u> > r iff. F is zero at every point x of X with dim x > r.

DEFINITION (5.3): X has <u>cohomological dimension</u> $\leq$ d (write cd X $\leq$ d) iff. $H^q(X, F) = 0$ for all torsion sheaves F and all q > d.

THEOREM (5.4): Let X/k be a noetherian integral prescheme, k a separably algebraically closed field. Assume the field R(X) of rational functions of X has transcendence degree n over k . If F is a torsion sheaf on X, zero in dimension > r, then $H^q(X; F) = 0$ for $q > 2r$. In particular, cd $X \leq 2n$.

Proof: We may assume case (f). The assertion is trivial for $r = -1$ because a sheaf which is zero at all points is the zero sheaf (apply Proposition (1.8) to all $U \in \text{Cat } T_X$). Moreover the theorem is constant for $r \geq n$. We use induction on r. Assume the theorem true for $r = s - 1$ and let F be a torsion sheaf on X, zero in dimension > s. Let $\{i_\nu : x_\nu \longrightarrow X\}$ be the set of points of X of dimension s, and let $X_\nu = \overline{x}_\nu$ be the closure of x_ν with its canonical reduced structure. X_ν satisfies the conditions of the theorem if s is substituted for n, and the induction assumption implies that the theorem is true for X_ν with $r < s$. Consider the map $F \longrightarrow i_{\nu *} i_\nu^* F$. Since X is noetherian and F is zero in dimension > s it is easy to see that a section $\xi \in F(U)$ is mapped to zero in $i_{\nu *} i_\nu^* F$ for all but a finite number of ν. Hence one obtains a map $0 \longrightarrow K \longrightarrow F \longrightarrow \bigoplus_\nu i_{\nu *} i_\nu^* F \longrightarrow C \longrightarrow 0$ (where K and C are the kernel and cokernel respectively). All these sheaves are torsion, and F, $\bigoplus i_{\nu *} i_\nu^* F$ have the same stalks in dimension > s - 1. Therefore K and C are zero in dimension > s - 1. Applying the induction assumption and examining the relations in the cohomology, one reduces to the case $F = \bigoplus i_{\nu *} i_\nu^* F$, hence by II (5.5) to the case $F = i_{\nu *} i_\nu^* F$. This sheaf is zero outside of X_ν. Replacing X_ν by X and s by n we are reduced to the case $F = i_* F_0$

where $i : x \longrightarrow X$ is the general point of X and F_0 is a torsion sheaf on $x = \mathrm{Spec}\ R(X)$. Moreover we may assume the theorem true for $r < n$.

LEMMA (5.5): $R^q i_* F_0$ is zero in dimension $> n - q$.

Assume the lemma, and examine the spectral sequence (cf. II (4.11)) $E_2^{p,q} = H^p(X ; R^q i_* F_0) \Longrightarrow H^*(x ; F_0)$. We are interested in the terms $E_2^{p,0} = H^p(X ; i_* F_0)$. Now cohomology over a field is equivalent with Galois cohomology, so we can apply the dimension theory for Galois cohomology to the ending $H^q(x ; F_0)$ of the spectral sequence. Since tr. $\deg._k R(X) = n$, it follows that $H^q(x, F_0) = 0$ for $q > n$ (cf. Sem. Bourb. # 189, p. 12, Remarque). On the other hand, we are assuming the theorem for $r < n$, and so it follows from the lemma and Proposition (5.1) that $E_2^{p,q} = 0$ for $p > 2(n - q)$. The theorem can now be obtained by examination of the spectral sequence.

<u>Proof of</u> (5.5) : Let $y \in X$ be given, $\dim y = m > n - q$ and let $\epsilon : P \longrightarrow X$ be a geometric point centered at y. We have to show the stalk $(R^q i_* F_0)_P$ is zero. Let A be the hensel ring obtained in Lemma (4.10) and let $A_1, \cdots, A_z$ be the affine rings of the irreducible components of Spec A. The rings A_j are hensel rings by (4.4) (i). Let R_j be the field of quotients of A_j (A is reduced). R_j is immediately seen to be (separably) algebraic over

$R(X)$, and, applying Corollary 3.9 (or well known facts about Galois cohomology) one finds $(R^q i_* F_0)_{\bar{P}} \approx \bigoplus_j H^q(\mathrm{Spec}\ R_j\, ;\, F_j)$ where F_j is some (torsion) sheaf induced by F_0. Since R_j is algebraic over $R(X)$, tr. deg. ${}_k R_j = n$. Let K be the common residue field of A, A_j, so that K is the separable algebraic closure of k(y). We have tr. deg.$_k$ K = m. Now an equicharacteristic hensel ring contains fields over which the residue field is purely inseparable (this is easy to see). Hence R_j contains a separably algebraically closed field K_j with tr. deg.$_k$ K_j = m. So tr. deg.$_{K_j}$ R_j = n - m. The lemma now follows from the dimension theory for Galois cohomology.

COROLLARY (5.6): Let X/k be a noetherian prescheme, k a field, and let $X_1, \cdots, X_r$ be the irreducible components of X with their reduced structure. Then

$$\mathrm{cd}\, X \leq \mathrm{cd}\, k + 2 \max \{ \mathrm{tr.\ deg.}_k\, R(X_j) \} .$$

Applying dimension theory for Galois cohomology and spectral sequence (4.8) one reduces to the case k separably algebraically closed. Then the corollary follows by induction from (5.4) and the map $F \longrightarrow \bigoplus_{i}^{r} F_i$ where F_i is the sheaf induced on X_i (cf. also SGA I, Theorem 8.3).

CHAPTER IV. Calculations for curves and surfaces

Section 1. We assume throughout this chapter that we are in case (l) or case (f) and that X is quasi compact and quasi separated in case (f).

Let $X \longrightarrow Z$ be a morphism of preschemes, and A a group scheme over Z. Because of III (1.4), the functor $\mathrm{Hom}_Z(\ \ , A)$: Cat $T_X \longrightarrow (Ab)$ is a sheaf on X. We will denote such a sheaf by A_X, or merely A if it will not cause confusion. Of course, $A_X \approx \mathrm{Hom}_X(\ \ , A \underset{Z}{\times} X)$, It follows immediately from III (3.7), if A is separated and finitely presented over Z, that under the assumptions of III, Corollary (3.9) the sheaf $\mathrm{Hom}_Z(\ \ , A)$ commutes with the limit, i.e., that, denoting by F_i the sheaf induced by A_X on X_i ($F_i \approx A_{X_i}$) we have $\underrightarrow{F} \approx \underleftarrow{A_X}$ (notation of III Corollary (3.9)). This is also true in the case A is discrete (below).

Case A is discrete: An abelian group A determines canonically a group scheme over spec $\mathbb{Z}$, namely $\coprod_{|A|}$ spec $\mathbb{Z}$ with the obvious group law. We denote this scheme also by A, and hence have defined A_X. This sheaf can also be obtained in other ways. In fact, A_X is isomorphic to the associated sheaf of the constant presheaf whose group of sections on each $U \in$ Cat T_X is A. Therefore we call A_X a constant sheaf. Or, denoting by $i: \{X\} \longrightarrow$ Cat T_X the inclusion of the discrete topology $\{X\}$ (cf. I (2.4)), $A_X \approx i_s A = (i_p A)^{\#}$. In particular, the notation $\mathbb{Z}_X$ does not conflict with II (1.7) (however $\mathbb{Z} \neq$ spec $\mathbb{Z}$). Finally, suppose $U \in \text{Cat } T_X$ consists of finitely many (say r) connected components. Then $A_X(U) \approx A^r$.

If X is noetherian, integral, and normal, and if $i : x \longrightarrow X$ is the general point, then $i_* A_x \approx A_X$. This follows easily from SGAI, Prop. 10.1. But, it is false in general. For instance, for a nodal rational curve X, one gets $0 \longrightarrow A_X \longrightarrow i_* A_x \longrightarrow A_Q \longrightarrow 0$ where A_Q is the constant sheaf at the node Q (extended by zero).

Among nondiscrete group schemes, we will be particularly interested in the sheaf $(\mathbb{G}_m)_X$ of units of $\Gamma(U, \underset{\sim}{\mathcal{O}}_U)$ ($U \in \text{Cat } T_X$). (Here $\mathbb{G}_m = \text{spec } \mathbb{Z}[t, 1/t]$), and the sheaf of nth roots of unity μ_n. Suppose n is prime to char k(x) for all points $x \in X$. Then the nth power map $\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$ yields

(1.1) $$0 \longrightarrow \mu_{n_X} \longrightarrow \mathbb{G}_{m_X} \xrightarrow{n} \mathbb{G}_{m_X} \longrightarrow 0$$ (n prime to the residue char.)

In fact, this sequence is always left exact, and under the assumptions, extraction of an nth root of a unit gives an étale covering. Hence the cokernel is zero, as a sheaf.

PROPOSITION (1.2): $H^1(X, \mathbb{G}_m) \approx \text{Pic } X$.

<u>Proof</u>: Assume first X affine. Then we may assume we are in case (f). By II (3.6), H^1 may be computed as Czech cohomology. Since for a covering $\{U_\alpha \longrightarrow X\}$ in case (f) $\coprod U_\alpha \longrightarrow X$ is faithfully

flat and quasi compact, the result follows from descent theory (cf. Sem. Bourb. No. 190.4.e) and the fact that the topology contains sufficiently fine Zariski open coverings. For the general case, let T^Z be the Zariski topology on X, T^G the Grothendieck topology (case (1)), and $f : T^Z \longrightarrow T^G$ the inclusion. Knowing the result for affines, one finds $R^1 f^s \mathbb{G}_m = 0$, and so we are done by the Leray spectral sequence II (4.11).

Suppose now X is integral, and let $i : x \longrightarrow X$ be the general point (x = spec $R(X)$, $R(X)$ the field of rational function of X). We have $H^1(x, G_m) = 0$ by (1.2) (this is "Hilbert theorem 90"). More generally, $H^1(u, \mathbb{G}_m) = 0$ for all $u \in \text{Cat}\, T_x$. Hence $R^1 i_* \mathbb{G}_{m,x} = R^2 i_* (\mathbb{G}_m)_x = 0$. By II (4.11) we find

(1.3) $$H^1(X, i_* \mathbb{G}_m) = 0 \qquad (i : x \longrightarrow X \text{ the gen. pt., } X \text{ integral}).$$

There is a natural inclusion $(\mathbb{G}_m)_X \hookrightarrow i_* (\mathbb{G}_m)_x$. We write

(1.4) $$0 \longrightarrow \mathbb{G}_m \longrightarrow i_* \mathbb{G}_m \longrightarrow D_X \longrightarrow 0 \qquad (\text{" "}).$$

Here D_X is the sheaf of "Cartier divisors" on X. Combining (1.3), (1.4) one gets

(1.5) $$R(X) \longrightarrow H^0(X, D_X) \longrightarrow \text{Pic } X \longrightarrow 0 \qquad (X \text{ integral}).$$

It is clear from the "five lemma" that $H^0(X, D_X)$ is the same as it would be if D_X were defined by (1.4) in the Zariski topology (because $H^1(X, i_* \mathbb{G}_m) = 0$ also in the Zariski topology).

If X is moreover noetherian and regular, D_X is easily identified as

(1.6) $$D_X \approx \bigoplus_{\nu} i_{\nu *} \mathbb{Z}_{x_\nu}$$ (X noeth., reg.)

where $\{i_\nu : x_\nu \longrightarrow X\}$ are the general points of the irreducible closed subsets of codim 1. Since in the Tate cohomology $H^1(G, \mathbb{Z}) = 0$ for any G one finds $H^1(x_\nu, \mathbb{Z}) = R^1 i_{\nu *} \mathbb{Z} = 0$. Therefore by the Leray spectral sequence and II (5.5) (T_X is noetherian)

(1.7) $$H^1(X, D_X) = 0$$ (X noeth., reg.)

We get from (1.4) and the fact that $R^1 i_* (\mathbb{G}_m)_x = 0$,

(1.8) $$H^2(X, \mathbb{G}_m) \hookrightarrow H^2(x, \mathbb{G}_m)$$ (" ")

Presumably, if X is affine, $H^2(X, \mathbb{G}_m)$ is closely related to the Brauer group of its ring (cf. Auslander & Goldman (AG)).

Section 2. Case of an algebraic curve.

Let X be a noetherian integral scheme over a separably algebraically closed field k_0 and suppose the field $R(X)$ of rational functions of X has transcendence degree 1 over k_0. In particular, X could be an algebraic curve. Let $i : x \longrightarrow X$ be the general point. We have $H^q(x, \mathbb{G}_m) = 0$ for $q \geq 2$ by dimension theory for Galois cohomology, for $q = 1$ by (1.2) and for $q = 2$ by Tsen's theorem (cf. (L)) if k_0 is algebraically closed. In general, $H^2(x, \mathbb{G}_m)$ is anyhow a p-group ($p = \text{char } k_0$) as is seen from (1.1). This being true for any algebraic extension of $R(X)$, one finds $R^q i_* \mathbb{G}_m$ are p-groups for $q > 0$, and hence

$$(2.1) \qquad H^q(X, i_* \mathbb{G}_m) = 0 \qquad \text{(ignoring p-torsion)}$$

If X is a complete algebraic curve, so that $H^0(X, \mathbb{G}_m) \approx k_0^*$ (which is divisible by n if $(n, p) = 1$), one finds from (1.1)

$$(2.2) \qquad H^q(X, \mu_n) = \begin{cases} \mu_n & q = 0 \\ (\text{Pic } X)_n & q = 1 \\ (\text{Pic } X)/n \approx \mathbb{Z}/n & q = 2 \\ 0 & q > 2 \end{cases} \quad \begin{pmatrix} X \text{ complete,} \\ (n, p) = 1 \end{pmatrix}.$$

where $(\text{Pic } X)_n$ is the group of points of order n on the Jacobian of X. These are the expected values (However, one cannot hope for "good" values of $H^q(X, \mathbb{Z})$ since for instance $H^1(X, \mathbb{Z}) = 0$ if X is regular.). Note that in (2.2) the removal of a point of X has the effect of killing the H^2.

If X is regular, so D_X torsion free, one deduces from (1.1) and (1.4)

$$(2.3) \qquad 0 \longrightarrow \mu_n \longrightarrow i_* \mathbb{G}_m \longrightarrow i_* \mathbb{G}_m \longrightarrow D_X/nD_X \longrightarrow 0 \quad ((n, p) = 1)$$

which is an acyclic resolution of μ_n (possible p-torsion does not matter).

So one may write

$$(2.4) \qquad H^1(X, \mu_n) \quad \{f \in R(X)^* \mid (f) \equiv 0(n)\} \Big/ \{f \in (R(X)^*)^n\} \qquad ((n, p) = 1)$$

where (f) denotes the division of f.

Let again $i : x \longrightarrow X$ be the general point of X, and F any sheaf on x. Clearly $R^q i_* F$ is zero at $x (q > 0)$, hence is cohomology vanishes in dimension > 0 by III (5.4). Therefore the Leray spectral sequence $H^p(X, R^q i_* F) \Longrightarrow H^*(x, F)$ reduces to an exact sequence

$$(2.5) \qquad \cdots \longrightarrow H^q(X, i_* F) \longrightarrow H^q(x, F) \longrightarrow H^0(X, R^q i_* F) \longrightarrow \cdots \quad .$$

In fact, by the dimension theory for Galois cohomology, $H^p(x, F)$ and $R^q i_* F$ are zero for $q > 2$. If F is torsion or corresponds to a divisible $G(\overline{R(X)} / R(X))$ - module, they are zero for $q > 1$. The relevant part of (2.5) is then

$$(2.6) \qquad 0 \longrightarrow H^1(X, i_* F) \longrightarrow H^1(x, F) \longrightarrow H^0 R^1 \longrightarrow H^2(X, i_* F) \longrightarrow 0 .$$

Suppose X is regular. Then $H^0 R^1 \approx \bigoplus_p H^1(x_p, F_p)$ where p runs over closed points of X, x_p is the general point of the henselization of $\mathcal{O}_{X,p}$ and F_p is the induced sheaf of x_p. This is essentially the situation studied by Ogg (O) in the case $F = A_x$ is points with values in an abelian variety A/x. $H^1(X, i_* A_x)$ is then the group of "locally trivial principal homogeneous spaces" of A. Ogg derives a kind of duality between $H^0(X, i_*\hat{A})$ and $H^2(X, i_*A)$ ($\hat{A}$ the dual abelian variety). By similar methods one may obtain a perfect duality between $H^q(X, i_*F)$ and $H^{2-q}(X, i_*\hat{F})$ when F is points with values in a finite group scheme over x of dimension (as R(X)-module) prime to p and $\hat{F}$ is its Cartier dual. The duality is given by a cup product into $H^2(X, \mu) \approx \mathbb{Q}/\mathbb{Z}$ (prime to p), μ the sheaf of roots of unity.

Section 3. Local calculations in dimension 2.

Let k_0 be a separably algebraically closed field. We denote by $k_0\{x, y, \ldots, z\}$ the henselization of the local ring at $(0, \ldots, 0)$ of spec $k_0[x, y, \ldots, z]$. It is known that $k_0\{x, y, \ldots, z\}$ consists of those power series $\alpha \in k_0[[x, y, \ldots, z]]$ which are algebraic over $k[x, y, \ldots, z]$. We will generally denote the closed point of spec $k_0\{x, y, \ldots, z\}$ by Q.

In the case of one variable $k_0\{t\}$, set $k = k_0(\{t\})$ = field of fractions of $k_0\{t\}$. The only algebraic extension of k of degree prime to p = char k_0 is $k[t^{1/n}]$, i.e., if $G = G(\tilde{k}/k)$ is the Galois group of the separable algebraic closure, and if $\bar{G}$ is the Galois group of the maximal extension prime to p, one has $\bar{G} \approx (\hat{\mathbb{Z}})_{(\text{prime to p})}$ = inverse limit of $\mathbb{Z}/n$, $(n, p) = 1$, and the kernel of the map $G \longrightarrow \bar{G}$ is a p-group. (This isomorphism is not canonical.) One finds

$$(3.1) \qquad H^q(\operatorname{spec} k, \mu_n) = H^q(\bar{G}, \mu_n) \approx \begin{cases} \mu_n & q = 0 \\ \mathbb{Z}/n & q = 1 \\ 0 & q > 1 \end{cases} \qquad ((n, p) = 1).$$

Here spec $k = (\text{spec } k_0\{t\}) - \Omega$.

Consider the case of two variables $k_0\{t, x\}$, and let

$$R = (k_0\{t, x\})_t = k_0\{t, x\}[1/t].$$

We wish to calculate the cohomology of spec R = spec $k_0\{t, x\}$ - locus of $t = 0$. Now $R \supset k = k_0(\{t\})$. Set

$$\bar{R} = R \underset{k}{\otimes} \bar{k} = \varinjlim_{k'} R \underset{k}{\otimes} k'$$

when $\bar{k}$ is the separable algebraic closure of k, and k' runs over finite subextensions of $\bar{k}$. Choose such a k' and let t' be a local parameter of the normalization $\mathcal{O}'$ of $k_0\{t\}$ in k', so that $\mathcal{O}' \approx k_0\{t'\}$. Then it is easily seen that $R' = R \underset{k}{\otimes} k' \approx (k_0\{t', x\})_{t'}$, which is a ring similar to R. Now R (hence R') is integrally closed. Moreover every prime of R is maximal. Finally, R is noetherian and Pic R = 0. Hence R is a PID. Since $\bar{R}$ is a limit of such rings, Pic $\bar{R} = 0$ and $\bar{R}$ is integrally closed. Since $\bar{R}$ has transc. deg. 1 over $\bar{k}$, every prime of R is maximal. Actually, $\bar{R}$ is also noetherian, hence a PID. To show this one has to show every prime ideal finitely generated. This is equivalent to showing every prime is induced from some R', i.e., to showing the primes of R' split into finitely many primes in $\bar{R}$, which amounts to showing $R/\mathfrak{p}$ finite over k for all primes $\mathfrak{p}$ of R. Now $\mathfrak{p}$ is induced by some prime element $f \in k_0\{t, x\}$, t not dividing f. Write $k_0\{t, x\}$ as a limit of rings étale, localized from an algebra of finite type over $k_0\{t\}[x]$. We can find f in some such ring, say A. Then A/(f) is of finite type over

$k_0\{t\}$. Applying ZMT, $A/(f)$ is localized from a finite $k_0\{t\}$ - algebra. Hence $A/(f)$ is finite over $k_0\{t\}$ and hensel by III (4. 3)(ii) and (4. 4)(ii). Therefore $A/(f) \approx k_0\{t, x\}/(f)$ and we are done.

By (1. 2) and (2. 1)

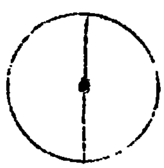
(3. 2) $$H^q(\text{spec } \bar{R}, \mathbb{G}_m) = 0, \; q > 0 \qquad \text{(ignore p).}$$

Now in a hensel ring over a separably algebraically closed field, the group of units is obviously divisible by n , (n, p) = 1. Hence for an R' as above, $R'^*/(R'^*)^n \approx \mathbb{Z}/n$, the generator being the residue of t'. Since $t^{1/n}$ is in $\bar{R}$, $\bar{R}^*$ is divisible by n. Therefore (1. 1) and (3. 2) yield

(3. 3) $$H^q(\text{spec } \bar{R}, \mu_n) = 0 \quad q > 0 \qquad ((n, p) = 1).$$

Applying the Hochschild-Serre spectral sequence III (4. 8):

(3. 4) $$H^q(\text{spec } R, \mathbb{G}_m) = 0, \; q > 0 \qquad \text{(ignore p)}$$



$$H^q(\text{spec } R, \mu_n) = \begin{cases} \mu_n, & q = 0 \\ \mathbb{Z}/n, & q = 1 \\ 0, & q > 1 \end{cases} \qquad ((n, p) = 1).$$

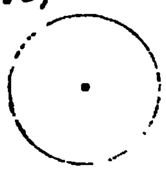
(The object on the left is a picture of spec R).

Set $Y = \text{spec } k_0\{t, x\} - C$ (C the closed point) and $j: \text{spec } R \longrightarrow Y$ the map. We have spec R = Y - X where $X = \text{spec } k_0(\{t\})$ is the locus t = 0.

One sees easily that $R^q j_* \mathbb{G}_m = 0$, $q > 0$ (ignore p). There is an exact sequence $0 \longrightarrow \mathbb{G}_{mY} \longrightarrow j_* \mathbb{G}_m \longrightarrow \mathbb{Z}_X \longrightarrow 0$, and hence $H^q(Y, \mathbb{G}_m) \approx H^{q-1}(X, \mathbb{Z})$ by (3.4), for $q > 1$. Clearly $H^1(Y, \mathbb{G}_m) = 0$. We have $H^{q-1}(X, \mathbb{Z}) \approx H^{q-2}(X, \mathbb{Q}/\mathbb{Z})$ because of the sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$, and so $H^2(X, \mathbb{Z}) \approx \widehat{\mathbb{Q}/\mathbb{Z}}$ (prime to p).

Therefore

(3.5)



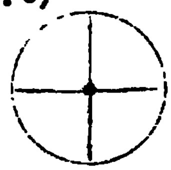
$$H^q(Y, \mu_n) \approx \begin{cases} \mu_n & q = 0 \\ 0 & q = 1 \\ \mathbb{Z}/n & q = 3 \end{cases} \qquad ((n, p) = 1).$$

This is the expected result if one thinks of Y as a 4 ball minus point.

We will also need to know the cohomology of spec S where $S = (k_0\{x, y\})_{xy} = k_0\{x, y\}[1/xy]$. This could be obtained from (3.4) but for our purposes the following method is better. Set $t = xy$, so that $S \supset k = k_0(\{t\})$, and set $\bar{S} = S \otimes_k \bar{k} = \varinjlim_{k'} S \otimes_k k'$ (k'/k finite separable). Here $S \otimes_k k'$ is not localized from a regular ring, but one can show anyhow that Pic $S \otimes_k k'$ is zero (cf. Note (4.8)). $\bar{S}$ is seen to be a PID by a similar argument to that used for $\bar{R}$ above, and therefore $H^q(\text{spec } \bar{S}, \mathbb{G}_m) = 0$, $q > 0$ (ignore p). However, $H^q(\text{spec } \bar{S}, \mathbb{G}_m) = 0$, $q > 0$ (ignore p). However, $H^0(\text{spec } \bar{S}, \mathbb{G}_m) = \bar{S}^*$ is not divisible. In fact, computing a little, one finds $\bar{S}^*/(\bar{S}^*)^n \approx (\mathbb{Z}/n \oplus \mathbb{Z}/n)/\Delta$ (Δ "diagonal", $(n, p) = 1$) with canonical generators $x \rightsquigarrow (1, 0)$ and $y \rightsquigarrow (0, 1)$. Hence $H^1(\text{spec } \bar{S}, \mu_n) = (\mathbb{Z}/n \oplus \mathbb{Z}/n) / \Delta$. x and y could also be viewed as generators of this group when written in the form (2.4).

Applying III (4.8),

(3.6)



$$H^q(\operatorname{spec} S, \mu_n) = \begin{cases} \mu_n & q = 0 \\ \mathbb{Z}/n \oplus \mathbb{Z}/n & q = 1 \\ \operatorname{Hom}(G, (\mathbb{Z}/n \oplus \mathbb{Z}/n)/\Delta) & q = 2 \end{cases} \qquad ((n,p)=1).$$

Section 4. <u>Digression on the Picard group of a curve over</u> $k_0\{t\}$.

Write $\mathcal{O} = k_0\{t\}$ (k_0 algebraically closed). In this section, V denotes a scheme proper over spec $\mathcal{O}$, which is irreducible and nonsingular, of dimension 2. We denote by U the general fibre of V, and we assume U is geometrically simple over $k = k_0(\{t\})$ (U is an algebraic curve). Finally, we assume V/spec $\mathcal{O}$ has a section. In this situation, if X denotes the closed fibre, then U is the open subscheme V - X of V. Let $X_i (i = 1, \ldots, m)$ be the irreducible reduced components of X, and say $X = \sum r_i X_i$ (cf. (4.1)(e)).

The scheme V/spec $\mathcal{O}$, being of finite type, can be obtained by a base extension from some V_0/spec $\mathcal{O}_0$ where $\mathcal{O}_0$ is a geometric discrete valuation ring over k_0 and $\mathcal{O}/\mathcal{O}_0$ is étale, and V_0 may be obtained by localization from an algebraic surface defined over k_0. The following properties, listed for the convenience of the reader, may be obtained with no difficulty from classical surface theory by applying such a descent, if they are not obvious anyhow.

(4.1) (a) Resolution of singularities if the assumption that V is nonsingular is dropped.

(b) V is projective over spec $\mathcal{O}$.

(c) Intersection theory for (Cartier) divisions on V, of the form $(D \cdot Y)$ (an integer) where D is any division and Y is a division with support on the closed fibre X. With this restriction, $(D \cdot Y)$ depends only on the division class of D. Hence the symbol may also be used if D denotes a division class.

(d) There exists a division class K, called canonical, appearing in the genus formula (e).

(e) Let $Y = \sum s_i X_i$ be a division with support on X, and assume $Y > 0$. One associates in the obvious way a closed subscheme of V to Y, which we denote by the same letter. The following formula holds:

$$1 - \chi(Y) = \frac{1}{2}((Y^2) + (K \cdot Y)) + 1 \overset{\text{defn}}{=} p(Y) \qquad (\chi \text{ the Euler Char.})$$

The definition of $p(Y)$ is extended formally to nonpositive Y.

(f) Castelnuovo's criterion for exceptional curves (of the first kind): $p(E) = 0$, $(E^2) = -1$ (E reduced, irreducible, with support on X), and the factorization of regular birational maps into locally quadratic transformations.

(g) If Y has support on X and $(Y^2) = 0$ then $Y = sX$ for some integer s (This follows from Hodge's theorem on a surface, and the fact that V has a section).

PROPOSITION (4.2): Consider the canonical map $\mathrm{Pic}\, V \longrightarrow \mathrm{Pic}\, X$.

(i) This map is surjective.

(ii) $\ker(\mathrm{Pic}\, V \longrightarrow \mathrm{Pic}\, X) = \mathcal{K}$ is uniquely divisible by n prime to char $k_0 = p$.

We omit the proof.

Denote by $\mathrm{Pic}^0 X$ the subgroup of Pic X consisting of divisor classes whose degree on each reduced, irreducible component X_i of X is zero, and by $\mathrm{Pic}^0 V$ the inverse image of $\mathrm{Pic}^0 X$ under the map $\mathrm{Pic}\, V \longrightarrow \mathrm{Pic}\, X$. $\mathrm{Pic}^0 V$ is the group of divisor classes D on V such that $(D \cdot X_i) = 0$ for all components X_i of X. Now $\mathrm{Pic}^0 X$ is easily seen to be divisible by n prime to p. Since obviously $\mathcal{K} = \ker(\mathrm{Pic}^0 V \longrightarrow \mathrm{Pic}^0 X)$ one finds by (4.2).

COROLLARY (4.3): (i) $\mathrm{Pic}^0 V$ is divisible by n

(ii) $(\mathrm{Pic}\, V)_n = (\mathrm{Pic}^0 V)_n \approx (\mathrm{Pic}^0 X)_n = (\mathrm{Pic}\, X)_n \qquad ((n, p) = 1)$,

where $(\ )_n$ denotes the subgroup of elements of order n.

Let $\mathcal{D}$ be the group of divisors (of the form $\sum s_i X_i$ with support on X, and $\bar{\mathcal{D}}$ its image in Pic V. By (4.1)(g), $\bar{\mathcal{D}} \approx \mathcal{D}/(X)$ where (X) is the subgroup generated by X. Clearly, U is the kernel of the map $\mathrm{Pic}\, V \longrightarrow \mathrm{Pic}\, U$. This map is surjective because V is nonsingular. Let A/spec k be the Jacobian of U, and for a field k'/k denote by $A_{k'}$ the points of A with values in k'. $A_k \subset \mathrm{Pic}$ is the subgroup of divisor classes of U of degree zero. Now by (4.1)(g), $\bar{\mathcal{D}} \cap \mathrm{Pic}^0 V = 0$. Hence the map Pic V $\longrightarrow$ Pic U induces an injection $\mathrm{Pic}^0\, V \hookrightarrow \mathrm{Pic}\, U$. Clearly its image is in A_k, and we denote this image by A_k^0.

LEMMA(4.4): A_k^0 depends only on U/k.

Proof: Given a curve geometrically simple, proper over k, one can imbed U as an open subscheme of some V/spec $\mathcal{O}$ as above, using (4.1) (a). Applying (4.1) (a) and (f), one reduces to showing $\mathrm{Pic}^0\, V \xrightarrow{\sim} \mathrm{Pic}^0\, V'$ when $f : V' \longrightarrow V$ is obtained by blowing up a point P of V. We have to show that if D' is a divisor on V' with $(D' \cdot X_i') = 0$ for all components X_i' of X', then D' is linearly equivalent to $f^{-1}(D)$ for some D with $(D \cdot X_i) = 0$ all i. This is easy.

PROPOSITION(4.5): A_k/A_k^0 is a finite group.

Proof: Consider the diagram

$$\begin{array}{ccccc}
\bar{\mathcal{D}} & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \epsilon \\
\uparrow & & \uparrow & & \uparrow \\
\bar{\mathcal{D}} & \longrightarrow & \mathrm{Pic}\, V & \longrightarrow & \mathrm{Pic}\, U \\
\uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathrm{Pic}^0\, V & \longrightarrow & A_k^0
\end{array}$$

where ϵ is the cokernel of $A_k^0 \longrightarrow \mathrm{Pic}\, U$. We claim the rows and columns are exact if zeros are added on the periphery. Here m is the number of irreducible components of X, and Pic $V \longrightarrow \mathbb{Z}^m$ is the map $D \rightsquigarrow ((D \cdot X_1), \cdots, (D \cdot X_m))$. This map may be factored as Pic $V \longrightarrow$ Pic $X \longrightarrow \mathbb{Z}^m$ where the second arrow is the "degree", and is surjective. Clearly therefore the middle column is exact. The only remaining one which is not obvious is the top row, but $\bar{\mathcal{B}} \longrightarrow \mathbb{Z}^m$ is injective because of (4.1) (g).

From the top row, since $\bar{\mathcal{B}}$ has rank $(m - 1)$ over $\mathbb{Z}$, ϵ is a group of finite type and of rank 1. But Pic $U/A_k \approx \mathbb{Z}$ is also of rank 1. Since we have $0 \longrightarrow A_k/A_k^0 \longrightarrow \epsilon \longrightarrow \text{Pic } U/A_k \longrightarrow 0$, the proposition follows.

Note that since A_k^0 is divisible if char $k_0 = 0$, one can characterize A_k^0 as the maximal divisible subgroup of A_k in that case.

Let k'/k be a finite separable extension, and let $\mathcal{O}'$ be the normalization of $\mathcal{O}$ in k', $U' = U \otimes_k k'$. Constructing a suitable $V'/\text{spec } \mathcal{O}'$ with general fibre U', one defines $A_{k'}^0 \subset A_{k'}$. By (4.1) (a), V' may be found with a map onto V (over $\mathcal{O}'/\mathcal{O}$). Therefore the map $A_k \hookrightarrow A_{k'}$ carries A_k^0 into $A_{k'}^0$. Set

$$A_{\bar{k}}^0 = \bigcup_{\bar{k} \supset k' \supset k} A_{k'}^0 \qquad (\bar{k} \text{ the sep. alg. clos. of } k,\ k'/k \text{ finite}),$$

so that $A_{\bar{k}}^0 \subset A_{\bar{k}}$. Denote by $(Z)_f$ the subgroup of an abelian group Z of elements of finite order prime to $p = \text{char } k_0$.

THEOREM(4.6): Suppose X reduced and with only ordinary nodes as singularities. Then

(a) $(A^0_{\bar{k}})_f = (A^0_k)_f$ (in other words, $G(\bar{k}/k)$ operates trivially on $(A^0_{\bar{k}})_f$.)

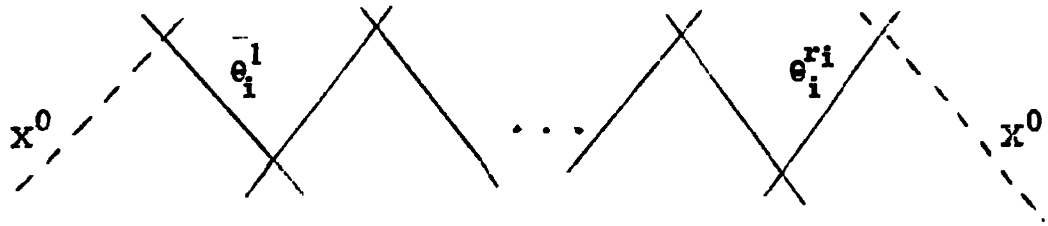
(b) If $a \in (A_{\bar{k}})_f$ and $\sigma \in G(\bar{k}/k)$ then $\sigma a - a \in A^0_k$.

Proof: Let $a \in (A^0_{\bar{k}})_f$, and let k'/k be a finite separable extension of k in which a becomes rational. Let $\mathcal{O}'$ be the normalization of $\mathcal{O}$ in k' and t' a local parameter of $\mathcal{O}'$. Then $\mathcal{O}' = \mathcal{O}[t']$. Set $V' = V \otimes_{\mathcal{O}} \mathcal{O}'$. V' is easily seen to be nonsingular except above the nodes of X. Above the nodes, V' is anyhow complete intersection, hence Cohen-Macauley, and therefore normal by Serre's criterion since there are no singular curves on V'. Again since V' is Cohen-Macauley, one sees immediately that the closed fibre X' of V' is reduced. The map $X' \longrightarrow X$ is 1 – 1, hence an isomorphism (because nodes are "minimal singularities").

Let $V'' \longrightarrow V'$ be obtained by resolving the singularity of V', and choose V'' minimal with this property. Let X'' be the closed fibre of V'' and $\theta_1, \cdots, \theta_s$ the connected components of the exceptional locus for the map $V'' \longrightarrow V'$, say $\theta_i = \theta_i^1 \cup \cdots \cup \theta_i^{r_i}$ where the θ_i^ν are the irreducible components. Denoting by X^0 the proper transform on V'' of the closed fibre X' of V', we have

$$X'' = X^0 + \sum_{i,\nu} a_i^\nu \, \theta_i^\nu ,$$

some $a_i^\nu > 0$. We claim $a_i^\nu = 1$, all i, ν, and that the curves θ_i consist of a configuration as depicted below (with suitable ordering of θ_i^ν)

(4.7) 

,

all intersections being transversal and θ_i^ν nonsingular and rational with $(\theta_i^{\nu\,2}) = -2$.

Note (4.8): This is of course purely a local statement, although the proof below utilizes the whole scheme V''. The configuration may be used to prove Pic $\bar{\bar{S}} = 0$ where $\bar{\bar{S}}$ is the ring used for (3.6).

These assertions may be verified by routine calculation: Note first that if $X'' - Z > 0$ ($Z > 0$) and if Z contains no exceptional curve then $p(X'' - Z) < p(X'')$. For,

$$p(X'' - Z) = \frac{1}{2}\Big((X''^2) + (Z^2) - 2(X'' \cdot Z) - (K \cdot X'') - (K \cdot Z)\Big) + 1$$

$$= p(X'') + \frac{1}{2}\Big((Z^2) - (K \cdot Z)\Big).$$

By (4.1) (g) and the fact that X'' has components with multiplicity 1, $(Z^2) < 0$, and $(K \cdot Z) > 0$ because Z contains no exceptional curve.

Therefore, since V'' was chosen to be minimal over V',

$$p(X'') > p\left(X^0 + \sum_{i,v} \theta_i^{\nu}\right)$$

with equality iff. all a_i^{ν} are equal 1. Calculating with (4.1) (e),

$$(4.9) \qquad p\left(X^0 + \sum_{i,\nu} \theta_i^{\nu}\right) = p(X^0) + \sum_{i,\nu} p(\theta_i^{\nu}) + \sum_{i,\nu} (X^0 \cdot \theta_i^{\nu}) + \sum_i \left(\sum_{\nu<\mu} (\theta_i^{\nu} \cdot \theta_i^{\mu})\right) - \sum_{i=1}^{s} r_i \ .$$

Here $p(\theta_i^{\nu}) \geq 0$, $\sum_{i,\nu} (X^0 \cdot \theta_i^{\nu}) \geq 2s$ since θ_i lies above a node of X', and

$$\sum_{\nu<\mu} (\theta_i^{\nu} \cdot \theta_i^{\mu}) \geq r_i - 1$$

since θ_i is connected. Moreover, $p(X^0) \geq p(X') - s$ since separation of a node reduces p by 1 (s is at most equal to the number of nodes). So combining,

$$p(X'') \geq p(X') - s + 2s + \sum_{i=1}^{s} r_i - 1 - \sum_{i=1}^{s} r_i = p(X') \ .$$

But by the invariance of Euler characteristic under specialization,

$p(X') = p(U')$ if U' is the general fibre of V'. Since U' is also the general fibre of V'', we have $p(X') = p(X'')$. Therefore all the above inequalities are equalities. Since $p(\theta_i^\nu) = 0$, $(\theta_i^{\nu 2})$ must be less than -1 by (4.1) (f). We have $(X'' \cdot \theta_i^\nu) = 0$ (since X'' is principal), hence $(X'' - \theta_i^\nu \cdot \theta_i^\nu) \geq 2$. But by the intersection numbers appearing in (4.9), the average value of $(X'' - \theta_i^\nu \cdot \theta_i^\nu)$ for i, ν varying is 2. Hence $(X'' - \theta_i^\nu \cdot \theta_i^\nu) = 2$ for all i, ν and so $(\theta_i^{\nu 2}) = -2$. It is now easily checked that (4.7) is the only configuration possible.

Returning to the proof of (4.6) (a), we had $a \in (A_k^0)_f$ rational over k'. This a corresponds to a unique division class $D'' \in (\mathrm{Pic}^0\, V'')_f$, so $(D'' \cdot X_j'') = 0$ for all components X_j'' of X''. In particular $(D'' \cdot \theta_i^\nu) = 0$ for all i, ν. Because the θ_i^ν are rational curves with $(\theta_i^{\nu 2}) = -2$, it is known that this implies the transform D' of D'' on V' is locally principal (cf. (C)). D' of course has finite order. Hence D' induces an element $d \in (\mathrm{Pic}\, X')_f \approx (\mathrm{Pic}\, X)_f$. By (4.3), there exists a $D \in (\mathrm{Pic}^0\, V)_f$ inducing d. One finds the transform of D on V'' is D'', so (a) is proved.

To prove (b), let $a \in (A_k)_f$ and choose k'/k as above, but a Galois extension, so that a is rational over k'. Retain the above notations. We may find a divisor D'' on V'' inducing the class a in $\mathrm{Pic}\, U'' = A_{k'}$. Say a is of order n. Then there is a rational function f on V'' such that

$$nD'' = (f) + \sum s_i X_i ,$$

some s_i (X_i'' the components of X''). Now $G = G(\bar{k}/k)$ operates on V'/V by automorphisms, and since V'' was chosen minimal over V', G also operates on V''/V by automorphisms (such a minimal model is unique). The components of X'' lifting from components of X are certainly left fixed by G. But, examining figure (4. 7), it is also clear that the θ_i^{ν}s must be left fixed. Hence X'' is left componentwise fixed. Obviously σa ($\sigma \in G$) is induced by the transformed divisor D''^{σ}, and we have

$$(nD'')^{\sigma} = n(D''^{\sigma}) = (f^{\sigma}) + \sum s_i X_i''^{\sigma} = (f^{\sigma}) + \sum s_i X_i'' .$$

Hence

$$n(D''^{\sigma} - D'') = (f^{\sigma}) - (f) = (f^{\sigma}/f)$$

so $D''^{\sigma} - D'' \in (\mathrm{Pic}^0 V'')_n$, and $\sigma a - a$ is therefore in A_k^0 by part (a).

Section 5. <u>Cohomology of a curve over</u> $k_0\{t\}$.

Let $V/\mathrm{spec}\,\mathcal{O}$, U, X be as in the previous section. We assume further that X is reduced, and has only nodes as singularities, and that U has positive genus (the case that U has genus zero is easier). We want to prove

(5. 1)

(i) $H^q(V, \mu_n) \xrightarrow{\sim} H^q(X, \mu_n)$, all q ((n, p) = 1)

(ii) $H^q(V, \mathbb{G}_m) = 0$ $q \geq 2$ (ignoring p-torsion).

The map for (i) is canonical, and is induced by the map $\mu_{n_V} \longrightarrow i_*(\mu_{n_X})$, where $i : X \longrightarrow V$.

Now $H^2(X, \mu_n) \approx \text{Pic } X/n \approx (\mathbb{Z}/n)^m$ (m = no. of comp. of X), and $H^q(X, \mu_n) = 0$ for $q > 2$. By (4.2), Pic $X/n \approx$ Pic V/n. Therefore, assuming (5.1) (i), the exact sequence (1.1) shows $H^q(V, \mathbb{G}_m)$ is torsion free, $q \geq 2$, prime to p. Hence (5.1) (ii) follows from (5.1) (i) once one knows $H^q(V, \mathbb{G}_m)$ is torsion for $q \geq 2$, and that fact can be checked easily from the exact sequence

$$0 \longrightarrow \mathbb{G}_{m_V} \longrightarrow j_* \mathbb{G}_{m_U} \longrightarrow D(X) \longrightarrow 0$$

($j : U \longrightarrow V$ the injection and $D(X)$ the sheaf of divisors on V with support on X). So we concentrate on (5.1) (i).

The isomorphism is obvious for $q = 0$. For $q = 1$, we have from (1.1)

$$0 \longrightarrow \mathcal{O}^*/(\mathcal{O}^*)^n \longrightarrow H^1(V, \mu_n) \longrightarrow (\text{Pic } V)_n \longrightarrow 0.$$

Since $\mathcal{O}$ is a Hensel ring, $\mathcal{O}^*$ is divisible by n. Hence $H^1(V, \mu_n) \approx (\text{Pic } V)_n$. Similarly, $H^1(X, \mu_n) \approx (\text{Pic } X)_n$. So by (4.3) (ii) (and trivial functorality) the assertion is true for $q = 1$.

The case q = 2 is harder. We use the relative cohomology sequence III (2.11) for the map $i : X \longrightarrow V$, and (1.1), to obtain an exact commutative diagram

$$
\begin{array}{ccccccccc}
 & & H^1_X(\mathbb{G}_m)/n & \longrightarrow & H^2_X(\mu_n) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Pic} V/n & \longrightarrow & H^2(V, \mu_n) & \longrightarrow & \epsilon & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Pic} U/n & \longrightarrow & H^2(U, \mu_n) & \longrightarrow & \delta & \longrightarrow & 0
\end{array} \tag{5.2}
$$

where ϵ, δ are the cokernels. The top terms are easily calculated by using III (2.8) and the spectral sequence $H^p(X, R^q i^! F) \Longrightarrow H^*_X(F)$. One finds $H^1_X(\mathbb{G}_m) \approx \mathbb{Z}^m$ and $H^2_X(\mu_n) \approx (\mathbb{Z}/n)^m$ (m = no. of comp. of X). Hence the top arrow of (5.2) is an isomorphism. Therefore $\epsilon \longrightarrow \delta$ is an injection.

Now if V' is an étale covering of V, and U', X' are the fibres of V', it is clear that the diagram (5.2) is mapped commutatively into the similar one for V', U', X'.

LEMMA (5.3): The image of $H^2(V, \mu_n)$ in $H^2(U, \mu_n)$ is annihilated in some étale covering V' of V.

Assume the lemma and consider the Hochschild-Serre spectral sequence III (4. 7) with $G = \pi_1(V)$. From the map $\mu_{n_V} \longrightarrow i_*(\mu_{n_X})$ one gets a morphism of spectral sequences with ending $H^*(V, \mu_n) \longrightarrow H^*(V, i_*\mu_n) \approx H^*(X, \mu_n)$. We have, in the notation of III (4. 7), $H^q(T_X, f(\bar{G}); F) \approx H^q(V', F')$ by II (4. 13), setting $V' = f(\bar{G})$ and F' the induced sheaf. Applying (5. 3) to the diagram (5. 2), one sees

$$\varinjlim_{\bar{G}} H^2(V', \mu_n) \approx \varinjlim_{\bar{G}} \operatorname{Pic} V'/n .$$

But by Proposition (4. 2),

$$\varinjlim \operatorname{Pic} V'/n \xrightarrow{\sim} \varinjlim \operatorname{Pic} X'/n .$$

Taking into account the isomorphisms in dimension 0, 1 one finds

$$\varinjlim H^q(V', \mu_n) \longrightarrow \varinjlim H^q(X', \mu_n) \qquad \text{for } q = 0, 1, 2.$$

(In fact, one gets 0 for $q = 1$, and 0 for $q = 2$ if X' contains no rational curves.) Therefore it follows from the spectral sequence that

$$H^q(V, \mu_n) \xrightarrow{\sim} H^q(X, \mu_n) \qquad \text{for } q = 0, 1, 2.$$

It remains to show $H^q(V, \mu_n) = 0$ for $q > 2$. (By III (5. 5) this is anyhow true if $q > 4$.) For this purpose, examine the Leray spectral sequence

$$(5.4) \qquad H^p(V, R^q j_* \mu_n) \Longrightarrow H^*(U, \mu_n) \qquad (j : U \longrightarrow V).$$

One finds, using (1.1), $R^1 j_* \mu_n \approx D(X)/n$ ($D(X)$ the divisors with support on X). If $\pi : \bar{X} \longrightarrow X$ is the normalization of X, it is easily seen that $D(X)/n \approx \pi_*(\mathbb{Z}/n)_{\bar{X}}$. Hence $H^p(V, R^1 j_* \mu_n) \approx H^p(\bar{X}, \mathbb{Z}/n)$ by III (4.11). This is computable by Section 2 since $\mathbb{Z}/n$ is isomorphic (not canonically) to μ_n. By III (5.5), $R^2 j_* \mu_n$ is concentrated at the closed points of X, and $R^q j_* \mu_n$ is concentrated at the nodes of X, and is computable there.

To calculate the encing $H^*(U, \mu_n)$ of (5.4), use the spectral sequence III (4.8). The nonzero terms of $E_2^{p,q}$ are (cf. 2.3))

$$(5.5) \qquad E_2^{p,q} = E_\infty^{p,q} = \left|\begin{array}{ll} \mathbb{Z}/n & H(G, \mathbb{Z}/n) \\ (A_k)_n & H(G, (A_{\bar{k}})_n) \\ \mu_n & H(G, \mu_n) \end{array}\right. \qquad (G = G(\bar{k}/k)).$$

Because of the special nature of G (cf. Section 3) one has a duality for the G-cohomology, due to Cgg ( ) from which, together with the autoduality of the Jacobian A, one finds the Euler characteristic

$$\chi(U, \mu_n) = \prod_q (-1)^q \ \# (H^q(U, \mu_n)) = 0$$

($\#$ (Z) = no. of elts. of Z).

$\chi(U, \mu_n)$ can also be calculated from the $E_2^{p,q}$ terms of (5.4), and the calculation shows that the unknown terms $\# (H^3(V, \mu_n))$ and $\# (H^4(V, \mu_n))$ are equal. So we are reduced to showing $H^4(V, \mu_n) = 0$.

Since $H^4(U, \mu_n) = 0$, one sees from (5.4) that the map $H^2R^1 \longrightarrow H^0R^4 = H^4(V, \mu_n)$ is surjective, so it would be convenient if H^2R^1 were zero. To achieve this, let V^- be obtained from V/σ by removing a section s_i passing through each component X_i of X. It is easy to see that $H^q(V, \mu_n) \xrightarrow{\sim} H^q(V^-, \mu_n)$ for $q \geq 3$ (apply 3.4) and III (4.9) to the injection $V^- \longrightarrow V$). But in the Leray spectral sequence similar to (5.4) for V^-, the term H^2R^1 is zero, so we are done.

Proof of Lemma (5.3): To begin with note that the assumption that X be reduced with only nodes is preserved in an étale covering V'/V. Set

$$Z = \text{im}(H^2(V, \mu_n) \longrightarrow H^2(U, \mu_n)).$$

From (5.5) we have a sequence

$$0 \longrightarrow H^1(G, (A_{\bar{K}})_n) \longrightarrow H^2(U, \mu_n) \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

where the $\mathbb{Z}/n$ is canonically isomorphic to Pic $\bar{U}/n$. This Pic $\bar{U}/n$ will be annihilated in a connected étale covering of degree n, hence in some V'/V if we know Pic V contains elements D of order n (as usual by adjoining $f^{1/n}$ for an f with $(f) = nD$). Such D are easily seen to exist by (4. 3) (ii) since X is reduced with only nodes and $p(X) > 0$. So for some V'/V the image of Z will be in the image of $H^1(G, (A)_n)$, which is contained in $H^1(G, (A')_n)$ (A' = Jacobian of U' and we are dropping the subscripts $\bar{k}$ on A, A').

Now since $(A^0)_n \approx (A^0_k)_n \approx (\mathrm{Pic}\ V)_n$ by (4. 6) (a) the elements of $(A^0)_n$ can be annihilated in some V'/V. Choose such a V' and let $\mathcal{A} \subset (A')_n$ be the image of $(A)_n$. We may assume $\mathcal{A} \cap (A'^0)_n = 0$. Now $\mathcal{A}$ is a G-submodule of $(A')_n$. Therefore, by (4. 6) (b) applied to $(A')_n$, G has trivial action on $\mathcal{A}$, and $H^1(G, \mathcal{A}) \approx \mathrm{Hom}(G, \mathcal{A}) \approx \mathcal{A}$ (not canon.). Applying (4. 6) (b) again, $H^1(G, \mathcal{A}) \subset H^1(G, (A')_n)$, hence the image of $H^1(G, (A)_n)$ is exactly $\mathrm{Hom}(G, \mathcal{A})$. We are going to show that the image of Z in $H^1(G, \mathcal{A})$ is zero at this stage.

Let $k_0\{x, y\}$ be the henselization of $\mathcal{O}'_{V,\Omega}$ at some closed point Ω of V (say Ω is a node of X) and let S be the ring $(k_0\{x, y\})_t$ ($t = xy$ if x and y are chosen suitably). We have a diagram

$$\begin{array}{ccc} U & \longleftarrow & \mathrm{spec}\ S \\ \downarrow & & \downarrow \\ V & \longleftarrow & \mathrm{spec}\ k_0\{x, y\} \end{array} \quad \text{and hence} \quad \begin{array}{ccc} H^2(U, \mu_n) & \longrightarrow & H^2(\mathrm{spec}\ S, \mu_n) \\ \uparrow & & \uparrow \\ H^2(V, \mu_n) & \longrightarrow & H^2(\mathrm{spec}\ k_0\{x, y\}, \mu_n) \\ & & \| \\ & & 0 \end{array}$$

Therefore Σ is mapped to zero in $H^2(\text{spec } S, \mu_n)$ for each node Q of X. Equivalently, the image of Σ in $H^2(S, \mu_n)$ must be zero for Q any node of X' (the closed fibre of V').

The reader will verify that there is a morphism of spectral sequences

$$\begin{array}{ccc} H^p(G, H^q(\bar{U}', \mu_n)) & \Longrightarrow & H^*(U', \mu_n) \\ \downarrow & & \downarrow \\ H^p(G, H^q(\text{spec } \bar{S}, \mu_n)) & \Longrightarrow & H^*(\text{spec } S, \mu_n) \end{array}$$

($\bar{U}' = U \underset{k}{\otimes} \bar{k}$, $\bar{S} = S \underset{k}{\otimes} \bar{k}$).

Applying (3.6) and the above discussion, one sees that in order to show the image of Σ in $H^1(G, (A')_n)$ is zero, it suffices to show that if $a \in \mathcal{A} \subset H(\bar{U}', \mu_n)$ and $a \neq 0$ then the image of a in $H^1(\bar{S}, \mu_n)$ is not zero for some node Q of X'.

Let D be a divisor on V' representing a in $(\text{Pic } U)_n$ and say D is chosen so as to avoid the nodes of X. Then there is a rational function f on V' so that $(f) = nD + \Sigma s_i X_i'$ for some $s_i \in \mathbb{Z}$, X_i' the components of X'. Here $\Sigma s_i X_i'$ is determined mod X' and mod n, but since $a \notin (A^0)_n$, $\Sigma s_i X_i' \not\equiv 0$ (mod X', n). Hence for some pair of indices, say for 1, 2, $s_1 \not\equiv s_2$ (mod n) and we may assume $X_1' \cap X_2' \neq \emptyset$.

Let Q be a point of intersection of X_1', X_2'. f represents a unit in the ring S obtained from Q, and in the notation of Section 3 (cf. (3.6)) this unit is not zero in $(\mathbb{Z}/n \oplus \mathbb{Z}/n)/\Delta$ since $s_1 \not\equiv s_2 \pmod n$. Obviously this implies that a does not have zero image in $H^1(\bar{S}, \mu_n)$ (cf. (2.5) if necessary), and we are done.

Section 6. Case of an algebraic surface. Let $\pi : V \longrightarrow C$ be a map of a complete nonsingular algebraic surface V onto a nonsingular curve C, everything defined over an algebraically closed field k_0. We assume that V/C has a section, that the general fibre of π is geometrically irreducible and simple, and that all fibres are reduced, with at most nodes as singularities. We want to get information about the cohomology of V in terms of π.

Given an algebraic surface V/k_0, it will in general be necessary to blow up a few points of V in order that such a map exist, so one should first examine the effect of a locally quadratic transformation $f : V' \longrightarrow V$ at a closed point Q of V on the cohomology. This is easily done, and one finds

(6.1)

$$R^q f_* \mathbb{G}_m = \begin{cases} \mathbb{G}_m & q = 0 \\ \mathbb{Z}_Q & q = 1 \\ 0 & q > 1 \end{cases} \qquad \text{(ignore p)}$$

$$R^q f_* \mu_n = \begin{cases} \mu_n & q = 0 \\ (\mathbb{Z}/n)_Q & q = 2 \\ 0 & q \neq 0, 2 \end{cases} \qquad ((n, p) = 1)$$

where $\mathbb{Z}_Q$ denotes the extension by zero of the sheaf $\mathbb{Z}$ on Q, and $(\mathbb{Z}/n)_Q$ is similar. Hence the only change in $H^q(V, \mathbb{G}_m)$ is in the picard group, where it is the obvious one.

Returning to the map π, we have $R^0\pi_* \mathbb{G}_m = \mathbb{G}_{m_C}$ since V/C is proper, and $R^1\pi_*\mathbb{G}_m = \underset{\sim}{\text{Pic}}$ V/C is the relative picard functor (by definition, cf. Sem.Bourb. #232) viewed as a sheaf on C. Since V/C has a section, H^0R^1 = Pic V/Pic C (cf. ibid). Passing to the limit to compute the stalks of $R^q\pi_*\mathbb{G}_m$, and applying (2. 2), (5. 1) (ii) one finds

$$(6.2) \qquad R^q\pi_*\mathbb{G}_m = \begin{cases} \mathbb{G}_m & q = 0 \\ \underset{\sim}{\text{Pic}}\ V/C & q = 1 \\ 0 & q > 1 \end{cases} \qquad \text{(ignore p)}.$$

Let $i : c \longrightarrow C$ be the general point, A/c the Jacobian of the general fibre U of π. If $F \subset \underset{\sim}{\text{Pic}}$ V/C denotes the subsheaf of divisor classes whose image on U is of degree zero, we get exact sequence

$$(6.3) \qquad \begin{array}{c} 0 \longrightarrow F \longrightarrow \underset{\sim}{\text{Pic}}\ V/C \longrightarrow \mathbb{Z} \longrightarrow 0 \\ \\ 0 \longrightarrow \epsilon \longrightarrow F \longrightarrow i_*A \longrightarrow 0 \end{array}$$

where ϵ is concentrated at the points of C whose fibre for π is reducible, and A is viewed as a sheaf on c. So $H^q(C, \epsilon) = 0$, $q > 0$, hence

$H^q(C, F) \approx H^q(C, i_* A)$, $q > 0$. Now $H^1(C, \mathbb{Z}) = 0$ (cf. Section 1). Moreover, since V/C has a section, the map $H^0(C, \underset{\sim}{Pic}\ V/C) \longrightarrow H^0(C, \mathbb{Z})$ obtained from the first sequence is surjective. Therefore $H^1(C, \underset{\sim}{Pic}\ V/C) \approx H^1(C, i_* A)$, whence by (6. 2)

$$(6.4) \qquad H^2(V, \mathbb{G}_m) \approx H^1(C, i_* A) \qquad \text{(ignore p).}$$

Applying (1. 1),

$$(6.5) \qquad 0 \longrightarrow Pic\ V/n \longrightarrow H^2(V, \mu_n) \longrightarrow)H^1(C, i_* A))_n \longrightarrow 0 . \quad ((n, p) = 1).$$

Note (6. 6): The image of Pic V in $H^2(V, \mu_n)$ is analogous to the part ρ of the classical b_2 (second betti number) generated by the algebraic cycles, hence $\rho_0 = b_2 - \rho$ is interpreted by (6. 5) in terms of "locally trivial principal homogeneous spaces" of A (cf. Section 2 and (O)). The sequence (6. 5) may be viewed as a clarification of the results of Igusa (I). In case V/C is a pencil of elliptic curves it is also closely related to certain aspects of the beautiful paper of Kodaira (K). In order to show that $H^2(V, \mu_n)$ has the expected value, it is at present necessary to use explicit calculations of a somewhat devious nature involving the group $H^1(C, i_* A)$. These calculations may be found in (I) and (O).

To get information about $H^q(V, \mu_n)$ for $q > 2$, it is more convenient to analyze the Leray spectral sequence

$$(6.7) \qquad H^p(V, R^q\pi_* \mu_n) \Longrightarrow H^*(V, \mu_n)$$

directly. For simplicity we make a further assumption on π, namely that all the fibres are <u>irreducible.</u> Then the sheaf $\in$ of (6. 3) is zero, and one sees immediately that $R^1\pi_* \mu_n \approx (i_* A)_n \approx i_*(A)_n$. Moreover, there is an injection $(\underset{\sim}{\mathrm{Pic}}\ V/C)/n \longrightarrow R^2\pi_* \mu_n$. By (6. 3), $(\underset{\sim}{\mathrm{Pic}}\ V/C)/n \approx \mathbb{Z}/n$. Examining the stalks of $R^2\pi_* \mu_n$ by means of (2. 2) and (5. 1) (i), one finds the injection is an isomorphism. Hence

$$(6.8) \qquad R^q\pi_* \mu_n = \begin{cases} \mu_n & q = 0 \\ i_*(A)_n & q = 1 \\ \mathbb{Z}/n & q = 2 \\ 0 & q > 2 \end{cases} \qquad ((n, p) = 1, \text{ and the fibres of } \pi \text{ irreducible})$$

Utilizing the duality for finite group schemes over c (cf. Section 2) and the autoduality of A, one finds the nonzero terms $E_2^{p,q}$ of (6. 7) are

$$(6.9) \qquad E_2^{p,q} = E_\infty^{p,q} = \begin{array}{|lll} \mathbb{Z}/n & \widehat{(\mathrm{Pic}\ C)_n} & \widehat{\mu_n} \\ (A_k)_n & H^1(C, i_*(A)_n) & \widehat{(A_k)_n} \\ \mu_n & (\mathrm{Pic}\ C)_n & \mathbb{Z}/n \\ \hline \end{array}$$

Here the transgressions $H^0R^1 \longrightarrow H^2R^0$ and $H^0R^2 \longrightarrow H^2R^1$ are zero. This is seen by comparing (6. 9) with the known values of the ending $H^q(V, \mu_n)$ (q = 1, 2) of (6. 7). For the second, use (6. 5) and the so called "Kummer sequence" $0 \longrightarrow i_*(A)_n \longrightarrow i_* A \xrightarrow{n} i_* A \longrightarrow 0$ which is exact if π has irreducible fibres.

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ÆRRATA

Page 25, line 12'	the $\vee$ over the H is illegible
Page 26	delete underlining except under remark and proof
Page 29	add a $\times$ on middle line and a U as follows : $W \underset{U}{\times} V \quad V \underset{U}{\times} V \quad W \quad \underset{U}{\times} \quad V$
Page 36, line 2'	Since i carries, not Since is carries
Page 61, line 8'	respect
Page 65, line 2	$\mathcal{B} \longrightarrow \mathcal{A}$
Page 67, bottom	$i_* i^* j_* j^*$ not $i_* j^* j_* j^*$
Page 69, line 3	delete comma
Page 71, line 3'	that i_* is exact
Page 73, line 7	replace k over the arrow by an h
Page 80, line 7'	a presheaf
line 4'	is flask
Page 88, line 11	$\bar{B} = B/$ (add bar)
line 9'	Nakayama
Page 89, line 7'	delete A at end of line
Page 91, line 3	$\bar{A}$ -algebras
line 5	$\bar{A}$ -algebra
line 12'	$\mathrm{Hom}_{\bar{A}}(\bar{B}, \bar{C})$
line 11'	$\bar{B} \otimes_{\bar{A}} \bar{C} \longrightarrow \bar{C}$
line 10'	$\bar{B} \otimes_{\bar{A}} \bar{C} \approx \bar{C} \times L$
Page 97, line 9'	with reference to k , instead of the
Page 98, line 2'	X_{ν}.
Page 103, line 10	$R^1 i_* (\mathbb{G}_m)_x$
Page 106, line 3	divisor, not division
line 5	its cohomology
Page 110,(3.5)	0 $q = 1, 2$
line 6'	delete However , ...
line 5'	delete (ignore p)
line 4'	add "the" at the end of the line

Page 113, line 7'	X_i)
Page 113, line 5'	replace U by $\bar{\mathcal{U}}$
line 2'	Pic U
Page 119, line 10	divisor not division
Page 120, line 6'	Let V/spec $\mathcal{V}$
Page 124, line 10	ending
line 4'	Ogg (0)
Page 94, line 7	For $R^q\mathrm{id}\, F$, read $R^q\mathrm{id}_* F$.
line 11	For the subscript p, read ^{p}
Page 118, line 2	For v, read ν