Recall $\mathcal{A}$ abelian category

$C^\cdot(\mathcal{A})$ (co)chain complexes over $\mathcal{A}$

$$D(\mathcal{A}) = C^\cdot(\mathcal{A}) \underbrace{[q\text{-iso}]^{-1}}$$

- $f : X^\cdot \to Y^\cdot$ is quasi-iso $\Leftrightarrow$ $H^i(f) : H^i(X^\cdot) \xrightarrow{\sim} H^i(Y^\cdot)$

Def $\mathcal{C} \in \mathrm{Cat}$ $\quad \underline{W \subset \mathrm{Mor}(\mathcal{C})}$

If $\exists\ \mathcal{C}[W^{-1}] \in \mathrm{Cat}$

$\quad + \ \mathcal{C} \xrightarrow{\gamma} \mathcal{C}[W^{-1}]$ functor

s.t. $\forall F : \mathcal{C} \to \mathcal{D}$

$\quad$ $\underline{\text{s.t. } F(W) \subset \mathrm{Iso}(\mathcal{D})}$

$\exists!\ F' : \mathcal{C}[W^{-1}] \to \mathcal{D}$ s.t. $\quad F = F' \circ \gamma$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[dr, "F"'] & \mathcal{C}[W^{-1}] \arrow[d, dashed, "F'"] \\ & \mathcal{D} \end{tikzcd}$$

$\mathcal{C}[W^{-1}]$ = $\underline{\text{localization}}$ of $\mathcal{C}$ % W

Lemma $\forall \mathcal{C}, \forall W$, $\mathcal{C}[W^{-1}]$ always exists

Pf/ $\mathrm{Obj}(\mathcal{C}[W^{-1}]) = \mathrm{Obj}(\mathcal{C})$ $\qquad (X \xrightarrow{f} Y) \mapsto \left(X \xleftarrow{=} X \xrightarrow{f} Y \right)$

$\nearrow$ $\mathrm{Obj}(\mathcal{C}[W^{-1}]) = \mathrm{Obj}(\mathcal{C})$

$$\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X,Y) = \left\{ \Gamma = \left(X = X_0 \leftarrow X_1 \rightarrow X_2 \leftarrow X_3 \rightarrow X_4 \cdots X_{2n-2} \leftarrow X_{2n-1} \rightarrow X_{2n} = Y \right) \right\} \big/ \sim$$

where $(X_{2i+1} \to X_{2i+2}) \in \mathrm{Mor}(\mathcal{C})$

$(X_{2i} \leftarrow X_{2i+1}) \in W \cup \{id\}$

where

$$\begin{tikzcd}
 & X_1 \arrow[r] \arrow[dl] \arrow[d] & X_2 \arrow[d] & \cdots \arrow[l] \arrow[r] & X_{2n-2} \arrow[d] & X_{2n-1} \arrow[l] \arrow[d] \arrow[dr] & \\
X_0 & X_1' \arrow[l] \arrow[r] & X_2' & \cdots \arrow[l] \arrow[r] & X_{2n-2}' & X_{2n-1}' \arrow[l] \arrow[r] & X_{2n}
\end{tikzcd}$$

$\Rightarrow \Gamma \sim \Gamma'$ $\square$.

<u>Pb</u> 1) $\mathcal{C}[W^{-1}]$ need not be small

2) Hard to compute Hom.

<u>Basic homotopical algebra</u> $\mathcal{C} \in \mathrm{Cat}$

<u>Def</u> 1) $X \xrightarrow{f} Y$, $U \xrightarrow{g} V$ $\in \mathrm{Mor}(\mathcal{C})$, we say that f is a <u>retract</u> of g

if $\exists$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, equal] & U \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\
Y \arrow[r] \arrow[rr, bend right, equal] & V \arrow[r] & Y
\end{tikzcd}$$

2) $X, U \in \mathcal{C}$ X is a <u>retract</u> of U if id_X is a retract of id_U.

<u>Def</u> $F \subset \mathrm{Mor}(\mathcal{C})$

1) F is stable under retracts if $\forall g \in F$, $\forall f$ retract of g, $f \in F$.

2) F — — — — pushouts if $\forall$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & U \arrow[d, "g"] \\ Y \arrow[r] & V \end{tikzcd}$$

pushout square, $f \in F \Rightarrow g \in F$

2') (pullbacks) (pullback square) $g \in F \Rightarrow f \in F$.

3) F — — — — transfinite compositions

if $\forall$ functor $X : I \to \mathcal{C}$ s.t.

1) I well-ordered with initial element 0

(I = ordinal)

2) $\forall i \in I \setminus \{0\}$, $\operatorname{colim}_{j<i} X(j)$ exists

and $\operatorname{colim}_{j<i} X(j) \to X(i) \in F$.

Then $\operatorname{colim}_{i \in I} X(i)$ exists, and $\left(X(0) \to \operatorname{colim}_{i \in I} X(i)\right) \in F$.

4) F is saturated if it satisfies 1) 2) 3)

5) F satisfies "2 out of 3" property if

$\forall$ $\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[rr, bend right, "h"'] & \cdot \arrow[r, "g"] & \cdot \end{tikzcd} \in \mathcal{C}$, 2 of $\{f, g, h\}$ are in F

$\Rightarrow$ so is the third

Rk If F contains all id's & satisfies 2) 3)

then F is stable under small sums

$$\coprod_{i\in I} X_i \xrightarrow{\coprod_{i\in I} u_i} \coprod_{i\in I} Y_i$$

Def $A \xrightarrow{i} B \in \mathcal{C}$

$X \xrightarrow{p} Y \in \mathcal{C}$

If $\forall$

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"'] \arrow[ur, dashed, "\exists h"] & Y \end{tikzcd}$$

$\exists h$ s.t. $hi = a$ & $ph = b$

We say - h is a filler

- i has the left lifting property % p (LLP)

- p has the right LP % i (RLP).

- $F \subset \mathrm{Mor}(\mathcal{C})$

$L(F) = \{ i \in \mathrm{Mor}(\mathcal{C}) \mid \forall f \in F,\ i \text{ has LLP \% } f \}$

$R(F) = \{ p \in \mathrm{Mor}(\mathcal{C}) \mid \forall f \in F,\ p \text{ has RLP \% } f \}$.

Lemma $F, G \in \mathrm{Mor}(\mathcal{C})$

- $F \subset R(G) \iff G \subset L(F)$

- $F \subset G \Rightarrow L(G) \subset L(F)$

$R(G) \subset R(F)$.

- $R(F) = R(L(R(F)))$

  $L(F) = L(R(L(F)))$

- $L(F)$ is saturated.

  $R(F)$ is cosaturated (i.e. saturated in $\mathcal{C}^{op}$).

<u>Lemma</u> (retract Lemma)

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & T \arrow[ur, "p"'] & \end{tikzcd} \in \mathcal{C} \qquad f = p\,i$$

1) $f \in R(i) \Rightarrow f$ is a retract of p

2) $f \in L(p) \Rightarrow$ - — — i

D/ 1)

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r, "p"'] \arrow[ur, dashed, "\exists h"] & Y \end{tikzcd} \Rightarrow \begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] \arrow[rr, bend left, no head] & T \arrow[r, "h"] \arrow[d, "p"] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

2) work in $\mathcal{C}^{op}$. $\square$.

<u>Ex</u> $\mathcal{C} = Set$ $\qquad \emptyset \xrightarrow{i} \{*\}$.

$R(i) = \{\text{surjections}\}$.

$L(R(i)) \overset{AC}{=} \{\text{injections}\}$ = smallest saturated class containing i.

<u>Def</u> A <u>weak factorization system</u> in $\mathcal{C}$ is $(A, B) \subset Mor(\mathcal{C})$

s.t. a) A & B are stable under retracts

b) $A \subset L(B)$

c) $\forall\, X \xrightarrow{f} Y \in \mathcal{C}$ $\exists$ factorization $f = pi$, $i \in A$, $p \in B$.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed, "i"'] & & Y \\ & T \arrow[ur, dashed, "p"'] & \end{tikzcd}$$

__Lemma__ $F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$ adjoint functors

(A, B) WFS, (A', B') WFS. Then $F(A) \subset A'$ $\iff$ $G(B') \subset B$.

__Recall__ κ cardinal, E = poset, $E \neq \emptyset$.

E is __κ-filtered__ if

$\forall\, J \in Set$, $\# J < \kappa$, $\forall\, (x_j)_{j \in J} \in E$

$\exists\, x \in E$, $\forall j \in J$, $x \geq x_j$

__Prop__ (small object argument)

- $\mathcal{C}$ = locally small cat. + small colimits
- $I \subset Mor(\mathcal{C})$ set of morphisms, κ cardinal, s.t.

  $\forall\, k \xrightarrow{i} l \in I$, $Hom_{\mathcal{C}}(k, -) : \mathcal{C} \to Set$

  commutes with colimits indexed by κ-filtered ordinals

Then
- $(L(R(I)), R(I))$ is a WFS
- $L(R(I))$ = smallest saturated class containing I.

D/ Hovey, Model categories, 2.1.14. P32.

Idea: $X \xrightarrow{f} Y \in \mathcal{C}$ $\quad \lambda = \kappa$-filtered ordinal

define $Z^f : \lambda \to \mathcal{C}$ s.t. $\quad$ - $Z^f_0 = X$ $\qquad Z^f_\lambda \to Z^f_{\lambda+1} = \amalg(\cdot)$

- f induces $Z^f \to Y$

- $E_f = \operatorname{colim} Z^f \to Y$

$X \nearrow$ $\qquad$ + transfinite induction.

$\square$.

<u>Cor</u> A = small cat, $\quad \mathcal{C} = \mathrm{Fun}(A^{op}, \mathrm{Set})$ presheaves

$I \subset \mathrm{Mor}(\mathcal{C})$ small set

Then $(L(R(I)), R(I))$ is a WFS on $\mathcal{C}$.

<u>Def</u> 1) A <u>(closed)</u> <u>model category</u>

= locally small category $\mathcal{C}$

+ 3 classes of morphisms W, Fib, Cof $\subset \mathrm{Mor}(\mathcal{C})$

s.t. i) $\mathcal{C}$ has finite limits/colimits

ii) W satisfies "2 out of 3" property

iii) (Cof, Fib $\cap W$) and (Cof $\cap W$, Fib) are WFS.

2) W = {weak equivalences}

$\mathrm{Fib} = \{ \text{fibrations} \}$

$\mathrm{Cof} = \{ \text{cofibrations} \}$.

$\mathrm{Fib} \cap W = \{ \text{trivial / acyclic fibrations} \}$

$\mathrm{Cof} \cap W = \{ \text{trivial / acyclic cofibrations} \}$

3) $\emptyset \in \mathcal{C}$ initial

$* \in \mathcal{C}$ final

$X \in \mathcal{C}$ <u>fibrant</u> $\Leftrightarrow$ $\begin{array}{c} X \\ \downarrow \\ * \end{array} \in \mathrm{Fib}$.

<u>cofibrant</u> $\Leftrightarrow$ $\begin{array}{c} \emptyset \\ \downarrow \\ X \end{array} \in \mathrm{Cof}$.

<u>Rk</u> 1) $\mathrm{Iso}(\mathcal{C}) \subset \mathrm{Fib} \cap \mathrm{Cof} \cap W$

2) $\forall\ X \xrightarrow{f} Y \in \mathcal{C}$ $\exists$

$$\begin{tikzcd} & \mathrm{Cof} \quad \underset{\sim}{P} \quad \mathrm{Fib} \cap W \arrow[dr, "D(f)"] & \\ X \arrow[ur, "C(f)"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{Cof} \cap W \quad \underset{\sim}{I} \quad \mathrm{Fib} \arrow[dr] & \\ X \arrow[ur] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Sometimes we require: – $\mathcal{C}$ has small limits / colimits

– functorial factorizations

(all examples satisfy these)

3) <u>W</u> is essential

Cof & Fib are auxillary.

4) Model category has redundancies:

<u>Lemma</u> $\mathrm{Cof} = L(\mathrm{Fib} \cap W)$

$\mathrm{Fib} = R(\mathrm{Cof} \cap W)$.

P/ Factorization + retract Lemma. $\square$.

<u>Ex</u> 1) $\mathcal{C} \in \mathrm{Cat}$ + finite lim, colim

- $W = \mathrm{Iso}$, $\mathrm{Fib} = \mathrm{Cof} = \mathrm{Mor}(\mathcal{C})$.

- $W = \mathrm{Fib} = \mathrm{Mor}(\mathcal{C})$, $\mathrm{Cof} = \mathrm{Iso}(\mathcal{C})$

- $W = \mathrm{Cof} = \mathrm{Mor}(\mathcal{C})$, $\mathrm{Fib} = \mathrm{Iso}(\mathcal{C})$

2) $A \in \mathrm{Cat}$ $\mathcal{C} = \mathrm{Fun}(A^{op}, \mathrm{Set})$ presheaves

$\mathcal{W} = \mathrm{Mor}(\mathcal{C})$, $\mathrm{Cof} = \mathrm{Mono}(\mathcal{C})$ $\mathrm{Fib} = R(\mathrm{Cof} \cap W)$.

P/ Let $I = \{ K \to L \in \mathrm{Mono}(\mathcal{C}) \mid L = \text{quotient of a representable} \}$

$AC \Rightarrow \mathrm{Mono}(\mathcal{C}) = L(R(I))$

$\overset{\text{small object argument}}{\Longrightarrow}$ $(\mathrm{Mono}(\mathcal{C}), R(I))$ WFS. $\square$.

<u>Lemma</u> $\mathcal{C}$ model cat.

1) $\mathcal{C}^{op}$ model cat: $W = \mathcal{W}(\mathcal{C})$

$\mathrm{Cof} = \mathrm{Fib}(\mathcal{C})$

$\mathrm{Fib} = \mathrm{Cof}(\mathcal{C})$

2) $\mathcal{C}/X$ model cat : $\quad Cof = Cof(\mathcal{C}) \cap Mor(\mathcal{C}/X)$

etc.

3) $X\backslash\mathcal{C}$ model cat.

4) Cof is stable under p.o.

Fib is stable under p.b.

5). Cof, Fib & W are closed under compositions.

<u>Prop</u> (Ken Brown's lemma)

- $\mathcal{C}$ model cat

- $\mathcal{D} \in Cat$, $\quad V \subset Mor(\mathcal{D})$ s.t. - $Iso(\mathcal{D}) \subset V$

w.e.

- V satisfies "2/3"

- $F : \mathcal{C} \to \mathcal{D}$ sends trivial cofibrations between cofibrant objects to V.

Then F sends w.e. - - - - - - to V.

Pf/ Let $X \xrightarrow{f} Y \in W$ $\quad X, Y$ cofibrant

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & Y \arrow[d, "j"] \arrow[dl, phantom, "\ulcorner"] & \in Cof \\ X \arrow[r, "i"] & X \amalg Y & \\ \in Cof & & \end{tikzcd}$$

$X \amalg Y \xrightarrow{(f, id_Y)} Y$ factors as

$$\begin{tikzcd} & T \arrow[dr, "p"] & \\ X \amalg Y \arrow[ur, "k"] & & Y \end{tikzcd}$$

$Cof \ni k$, $\quad p \in Fib \cap W$.

T cofibrant

$\in Cof \cap W$

$\in Cof \cap W$

$$\Rightarrow \quad \begin{tikzcd} X \arrow[r, "ki"', "\in Cof \cap W"] \arrow[d, "f"'] & T \arrow[dl, "p"] \\ Y & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "kj"', "\in Cof \cap W"] \arrow[d, equal] & T \arrow[dl, "p"] \\ Y & \end{tikzcd}$$

$$\Rightarrow \left.\begin{aligned} & F(X \xrightarrow{ki} T) \in V \\ & F(Y \xrightarrow{kj} T) \in V \\ & F(p) \in V \end{aligned}\right) \Rightarrow F(f) = F(pki) \in V. \quad \square.$$

Homotopy category

Def $\mathcal{C}$ model cat,

1). $A \in \mathcal{C}$

$$\begin{tikzcd} A \amalg A \arrow[r, "(\partial_0, \partial_1)"] \arrow[rr, bend left, "id_A \amalg id_A"] & IA \arrow[r, "\sigma"] & A \end{tikzcd}$$
$$\in Cof \qquad \in W$$

IA is a cylinder object of A

- $X \in \mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[rr, bend left, "(id_X, id_X)"] & X^I \arrow[r, "(d^0, d^1)"] & X \times X \end{tikzcd}$$
$$\in W \qquad \in Fib$$

X^I is a cocylinder / path object of X

2) $f_0, f_1 : A \longrightarrow X \in Mor(\mathcal{C})$

A <u>left homotopy</u> $f_0 \to f_1$ = $\left|\begin{array}{l} \text{cylinder object } IA \\ + \; h : IA \to X \end{array}\right.$

s.t. $i = 0, 1,$ $\quad h \partial_i = f_i$

<u>right homotopy</u> $f_0 \to f_1$ = $\left|\begin{array}{l} \text{cocylinder object } X^I \\ + \; k : A \to X^I \end{array}\right.$

s.t. $i = 0, 1,$ $\quad d^i k = f_i$.

<u>Lemma</u> <u>A cofibrant</u> <u>X fibrant</u> $\quad A \underset{f_1}{\overset{f_0}{\rightrightarrows}} X$. TFAE

1) $\exists$ left homotopy $f_0 \to f_1$

2) $\exists$ right homotopy $f_0 \to f_1$

3) $\forall IA$ cyl. obj, $\exists\, h : IA \to X$ s.t. $h \partial_i = f_i$

4) $\forall X^I$ cocyl. obj, $\exists k : A \to X^I$ s.t. $d^i k = f_i$

D/ suffices 1) ⟹ 4)

$A \xrightarrow[\;]{\partial_1 \,\in W} IA$

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] \arrow[ur, phantom, "\ulcorner"] & A \amalg A \end{tikzcd}$$

$\in Cof$

$\Rightarrow$ <u>$\partial_1 \in Cof \cap W$.</u>

similarly $\quad X^I \xrightarrow{(d^0, d^1)} X \times X$

$\in Fib$.

$\Rightarrow$

$$\begin{tikzcd} A \arrow[r, "sf_1"] \arrow[d, "\partial_1"'] & X^I \arrow[d, "(d^0, d^1)"] \\ IA \arrow[ur, dashed, "\exists K"] \arrow[r, "(h, f_1\sigma)"'] & X \times X \end{tikzcd}$$

put $k = K\partial_0$

$\Rightarrow d^1 k = d^1 K \partial_0 = f_1 \sigma \partial_0 = f_1 \, \mathrm{Id}_A = f_1$

$d^0 k = d^0 K \partial_0 = h \partial_0 = f_0$. $\square$

Lemma A cofibrant, X fibrant. Then "$\exists$ left htp" (or right htp) is an equivalence relation on $\mathrm{Hom}(A, X)$

Pf/ - Reflexive: clear

- Symmetry: follows from the equivalence 1) $\Rightarrow$ 3).

- Transitivity: $u, v, w \in \mathrm{Hom}(A, X)$

$$\begin{tikzcd} A \amalg A \arrow[r, "(\partial_0, \partial_1)"] \arrow[dr, "(\partial_0', \partial_1')"'] & IA \arrow[r, "\sigma"] & A \\ & I'A \arrow[ur, "\sigma'"'] & \end{tikzcd}$$

$h : IA \to X$

$h' : I'A \to X$

s.t. $\begin{cases} h\partial_0 = u \\ h\partial_1 = v = h'\partial_0' \\ h\partial_1' = w \end{cases}$

$$\begin{tikzcd} A \arrow[r, "\partial_1"] \arrow[d, "\partial_0'"'] & IA \arrow[d, "e"] \arrow[ddr, bend left, "\sigma"] & \\ I'A \arrow[r, "e'"'] & I''A \arrow[ul, phantom, "\ulcorner"] & \\ & & A \end{tikzcd}$$

$\exists ! \, \sigma'' : I''A \to A$

s.t. $\begin{cases} \sigma'' e = \sigma \\ \sigma'' e' = \sigma' \end{cases}$

$$\begin{tikzcd}
I'A \arrow[r, "e'"] \arrow[dr, "\sigma'"'] & I''A \arrow[d, dashed, "\exists!\sigma''"] \\
 & A
\end{tikzcd}$$

s.t. $\sigma'' e' = \sigma'$

$$\begin{tikzcd}
A \arrow[r, "(id, id)"] \arrow[d, "\partial_0'"'] & A \amalg A \arrow[r, "(\partial_0, \partial_1)", "\in Cof"'] \arrow[d, "id_A \amalg \partial_0'"', "\in Cof"] & IA \arrow[d, "e"] \\
I'A \arrow[r] & A \amalg I'A \arrow[r, "(e\partial_0, e')", "\in Cof"'] & I''A
\end{tikzcd}$$

$\Rightarrow (e\partial_0, e'\partial_1') \in Cof$

$\Rightarrow A \amalg A \xrightarrow{(e\partial_0, e'\partial_1')} I''A \xrightarrow{\sigma''} A$ cylinder object.

Define $h'' : I''A \to X$ s.t. $h''e = h$

$h''e' = h'$

$\Rightarrow h''\partial_0'' = u$

$h''\partial_1'' = w$. $\square$

<u>Def</u> $\mathcal{C}_c \subset \mathcal{C}$ full subcat of cofib objects

$\mathcal{C}_f \subset \mathcal{C}$ — — — fib objects

$\mathcal{C}_{cf} = \mathcal{C}_c \cap \mathcal{C}_f$

<u>Th</u>

$$\begin{tikzcd}
 & \mathcal{C}_c[W^{-1}] \arrow[dr, "\sim"] & \\
\mathcal{C}_{cf}[W^{-1}] \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & & \mathcal{C}[W^{-1}] \\
 & \mathcal{C}_f[W^{-1}] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

$$\mathcal{C}_f[W^{-1}]$$

D/ <u>Assume</u> $\exists$ <u>func. cofibrant replacement</u> Q

i.e. $\emptyset \underset{\in Cof}{\longrightarrow} QX \underset{\in Fib \cap W}{\xrightarrow{q_X}} X$

(The general case is true but harder)

Show that $\mathcal{C}_c[W^{-1}] \simeq \mathcal{C}[W^{-1}]$

$\mathcal{C}_c \xrightarrow{i} \mathcal{C}$ preserves $W \Rightarrow$ induces $\mathcal{C}_c[W^{-1}] \xrightarrow{i[W^{-1}]} \mathcal{C}[W^{-1}]$

$\Rightarrow Q : \mathcal{C} \to \mathcal{C}_c$ preserves $W \Rightarrow$ induces

$$Q[W^{-1}] : \mathcal{C}[W^{-1}] \to \mathcal{C}_c[W^{-1}].$$

$Q \circ i \xrightarrow{\in W} id_{\mathcal{C}_c}$ $\qquad$ $i[W^{-1}] \circ Q[W^{-1}] \simeq id_{\mathcal{C}_c[W^{-1}]}$

$\Rightarrow$

$i \circ Q \xrightarrow{\in W} id_{\mathcal{C}}$ $\qquad$ $Q[W^{-1}] \circ i[W^{-1}] \simeq id_{\mathcal{C}[W^{-1}]}$.

$\Rightarrow Q[W^{-1}]$ is an inverse of $i[W^{-1}]$. $\square$.

<u>Def</u> $[A, X] = \mathrm{Hom}_{\mathcal{C}}(A, X) /\sim_{\text{left homotopy}}$

$\Rightarrow [-,-] : \mathcal{C}_c^{op} \times \mathcal{C}_f \to \mathrm{Set}$.

$\Rightarrow \quad [-,-] : \underline{\mathcal{C}_c}^{op} \times \underline{\mathcal{C}_f} \to Set.$

<u>Prop</u> The functor $[-,-]$ preserves W

$\Rightarrow$ induces functor $[-,-] : \mathcal{C}_c^{op}[W^{-1}] \times \mathcal{C}_f[W^{-1}] \to Set$

P/ Let $A, B \in \mathcal{C}_c$, $A \xrightarrow{i} B \in W$, $X \in \mathcal{C}_f$

Need to show: $i^* : [B,X] \to [A,X]$ bijective.

Ken Brown's Lemma $\Rightarrow$ WMA $i \in Cof \cap W$

<u>Surjectivity</u>:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "\exists g"] & * \end{tikzcd}$$

$\forall f : A \to X$

$\exists g : B \to X \quad f = gi$

$\Rightarrow \quad i^*[g] = [f]$

<u>Injectivity</u> Let $f, g : B \to X$ s.t. $[fi] = [gi] \in [A,X]$

$\Rightarrow \exists \begin{cases} \text{cocylinder } X^I \xrightarrow{(d^0,d^1)} X \times X \\ \text{map } A \xrightarrow{k} X^I \end{cases}$ s.t. $d^0 k = fi$, $d^1 k = gi$

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "i"'] & X^I \arrow[d, "(d^0,d^1)"] \\ B \arrow[r, "(f,g)"'] \arrow[ur, dashed, "\exists K"] & X \times X \end{tikzcd}$$

$\Rightarrow [f] = [g] \in [B,X]$

Similarly, $\forall \begin{cases} A \in \mathcal{C}_c \\ X \xrightarrow{p} Y \in \mathcal{C}_f \cap W \end{cases}$ $p_* : [A,X] \xrightarrow{\sim} [A,Y]$

$\square$.

$\underline{\text{Th}}$ Let $\mathcal{C}_{cf}/\sim$ be the quotient category

Obj = cofibrant - fibrant objects in $\mathcal{C}$

$\mathrm{Hom}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)/\sim_{htp} = [A,B]$

Then $\exists$ iso of cat. $\underbrace{\mathcal{C}_{cf}/\sim}\ \simeq \underline{\mathcal{C}[W^{-1}]}$.

D/ Let $f, g : A \to X \in \mathcal{C}_{cf}$, $f \sim g$

$$\Rightarrow \exists \left| \begin{array}{l} \text{cylinder} \quad A \amalg A \xrightarrow{(\partial_0,\partial_1)} IA \underset{\in W}{\xrightarrow{\sigma}} A \\ h : A \to X \end{array} \right. \quad \text{s.t.} \quad \begin{array}{l} h\partial_0 = f \\ h\partial_1 = g. \end{array}$$

$$\left| \begin{array}{l} \sigma\partial_0 = \mathrm{Id}_A = \sigma\partial_1 \\ \sigma \in \mathrm{Iso}(\mathcal{C}[W^{-1}]) \end{array} \right. \Rightarrow [\partial_0] = [\partial_1] \in \mathcal{C}[W^{-1}]$$

$$\Rightarrow [f] = [g] \in \mathcal{C}[W^{-1}]$$

i.e. $\forall$ htp in $\mathcal{C}_{cf}$ induces an iso in $\mathcal{C}[W^{-1}]$

But $\forall A, B \in \mathcal{C}_{cf}$ $\quad \forall u : A \to B \in W$

$\forall X \in \mathcal{C}_{cf}$ $\quad \Rightarrow u^* : [B,X] \simeq [A,X]$

$\overset{\text{Yoneda}}{\Rightarrow} [u] \in \mathrm{Iso}(\underset{\sim\sim}{\mathcal{C}_{cf}/\sim})$

i.e. $\forall u : A \to B \in \mathcal{C}_{cf}$

$u \in \underset{\sim}{W} \iff u$ is a $\underline{\text{htp eq.}}$

$\Rightarrow \mathcal{C}_{cf}/\sim \ \simeq \mathcal{C}[W^{-1}]$. $\square$.

Cor $A \in \mathcal{C}_c$ $X \in \mathcal{C}_f$, $\exists$ iso of functors $[A, X] \simeq \mathrm{Hom}_{\mathcal{C}[W^{-1}]}(A, X)$

$$\mathcal{C}_c^{op}[W^{-1}] \times \mathcal{C}_f^{op}[W^{-1}] \longrightarrow \mathrm{Set}$$

Def $\underline{\mathcal{C}_{cf}/\sim} \simeq \mathcal{C}[W^{-1}]$ = the <u>homotopy category</u> of $\mathcal{C}$.

Cor $\mathcal{C}$ model cat, $\mathcal{C} \to \mathcal{C}[W^{-1}]$

- $\mathcal{C}$ locally small $\Rightarrow$ $\mathcal{C}[W^{-1}]$ locally small

- $f \in W \iff [f] \in \mathrm{Iso}(\mathcal{C}[W^{-1}])$

Derived functors

Def $\mathcal{C}$ = model cat

$\mathcal{D}$ = cat.

$F : \mathcal{C} \to \mathcal{D}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[d, "F"'] & \mathcal{C}[W^{-1}] \arrow[dl, "LF"'] \arrow[dl, bend left, "\Phi"] \arrow[d, phantom, "\Leftarrow"] \\ \mathcal{D} & \end{tikzcd}$$

1) A <u>left derived functor</u> of F is | functor $LF : \mathcal{C}[W^{-1}] \to \mathcal{D}$
| nat. trans $a_X : LF(\gamma(X)) \to F(X)$

s.t. $\forall \Phi : \mathcal{C}[W^{-1}] \to \mathcal{D}$

$\forall$ nat. trans $\alpha_X : \Phi(\gamma(X)) \longrightarrow F(X)$

$\exists !$ nat. morphism $f_Y : \Phi(Y) \to LF(Y)$.

s.t. $\alpha_X = a_X f_{\gamma(X)}$.

(i.e. LF = right Kan extension of F along γ)

2) A right derived functor of F is $\left|\begin{array}{l} RF : \mathcal{C}[W^{-1}] \to \mathcal{D} \\ b_x : F(x) \to RF(\gamma(x)) \end{array}\right.$

s.t. RF^{op} = Left derived functor of $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$.

Let $F : \mathcal{C} \to \mathcal{D}$ sends $\mathrm{Cof} \cap W$ in $\mathcal{C}_c$ to $\mathrm{Iso}(\mathcal{D})$

KBL $\Rightarrow$ F sends W in $\mathcal{C}_c$ to $\mathrm{Iso}(\mathcal{D})$

$$\Rightarrow \quad \begin{tikzcd} \mathcal{C}_c \arrow[r, "F|_{\mathcal{C}_c}"] \arrow[d] & \mathcal{D} \\ \mathcal{C}_c[W^{-1}] \arrow[ur, dashed, "\exists F_c"'] & \end{tikzcd} .$$

$$\begin{tikzcd} \mathcal{C}_c[W^{-1}] \arrow[r, shift left, "i"] & \mathcal{C}[W^{-1}] \arrow[l, shift left, "Q", "\sim"'] \end{tikzcd}$$

<u>Prop</u> $LF = F_c \circ Q \ : \mathcal{C}[W^{-1}] \to \mathcal{D}$

is a left derived functor of F.

<u>Cor</u> $\forall\ G : \mathcal{D} \to \mathcal{E}$, GLF is a left derived functor of GF.

<u>Def</u> $\mathcal{C}, \mathcal{C}'$ model cat, $\mathcal{C} \xrightarrow{\gamma} \mathcal{C}[W^{-1}]$

$$\mathcal{C}' \xrightarrow{\gamma'} \underbrace{\mathcal{C}'[W'^{-1}]}$$

- $F : \mathcal{C} \to \mathcal{C}'$ preserve $\mathrm{Cof} \cap W$

$\Rightarrow$ $\gamma' F$ sends $\mathrm{Cof} \cap W$ in $\mathcal{C}_c$ to $\mathrm{Iso}(\mathcal{C}'[W'^{-1}])$

$\Rightarrow$ $LF := L(\gamma' F) : \mathcal{C}[W^{-1}] \to \mathcal{C}'[W'^{-1}]$

<u>total LDF</u>

- $F : \mathcal{C} \to \mathcal{C}'$ preserve $\mathrm{Fib} \cap W$

$\Rightarrow$ $RF := R(\gamma' F) : \mathcal{C}[W^{-1}] \to \mathcal{C}'[W'^{-1}]$

<u>total RDF</u>

<u>Prop</u> $LF' \circ LF \simeq L(F' \circ F)$

$R(F' \circ F) \simeq RF' \circ RF.$

<u>Def</u> $\mathcal{C}, \mathcal{C}'$ model cat.

$F : \mathcal{C} \rightleftarrows \mathcal{C}' : G$ is a <u>Quillen adjunction</u>

if - F preserves Cof

- G preserves Fib.

$\Rightarrow$ F = <u>left Quillen functor</u>

G = <u>right — —</u>

<u>Lemma</u> (F, G) Quillen adj

$\iff$ F preserves Cof & Cof $\cap W$

$\iff$ G preserves Fib & Fib $\cap W$.

<u>Th</u> A Quillen adjunction (F, G)

induces an adjunction $LF : \mathcal{C}[W^{-1}] \rightleftarrows \mathcal{C}'[W'^{-1}] : RG$.

<u>Def</u> A Quillen adjunction (F, G) is a <u>Quillen equivalence</u>

if $\forall X \in \mathcal{C}_c$, $Y \in \mathcal{C}'_f$: $f : FX \to Y \in W' \iff \varphi(f) : X \to GY \in W$

<u>Th</u> (F, G) Quillen eq $\iff$ (LF, RG) is an adjoint equivalence

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<u>Examples</u>

<u>Top. spaces</u>

<u>Th</u> $\mathcal{C} = \mathrm{Top}$. $\exists$ model str.

W = weak htp. eq. (i.e. $f : X \to Y$ s.t. $\forall n$, $\pi_n(f)$ iso)

$\mathrm{Cof} = L\left(R\left(\begin{smallmatrix} S^{n-1} \\ \downarrow \\ D^n \end{smallmatrix}\right)\right)$ = retracts of relative cell complexes

$Cof = L(R(S^{n-1} \hookrightarrow D^n))$ = retracts of relative cell complexes

$$Fib = R\left(\begin{array}{ccc} D^n & & x \\ \downarrow & & \downarrow \\ D^n \times [0,1] & & (x,0) \end{array}\right)$$ = Serre fibrations.

$\rightsquigarrow Ho(Top)$.

<u>Chain complexes</u> R = ring $\mathcal{C} = C_\bullet(R)$

(unbounded) chain complexes

W = quasi-iso.

<u>Th</u> $\exists$ model structures on $C_\bullet(R)$

1) $Fib = \{$ degree-wise epimorphisms $\}$

$Cof = L(Fib \cap W)$

<u>(projective model structure)</u>

- $\mathcal{C}_f$ = all.

C (projectives)

$\cup$

- $\mathcal{C}_c$ = "dg-projective complexes"

$= \{ D \mid \forall A$ acyclic, $\underline{Hom}(D, A)$ acyclic $\}$.

$\cup$

C^- (projectives)

- X projective $\Leftrightarrow$ $\begin{cases} X \in \mathcal{C}_c \\ X \text{ acyclic.} \end{cases}$

- Cof = {degreewise split inclusion with cofibrant cokernel}

  Cof $\cap$ W $\subset$ injections with projective cokernel.

2) Cof = {degreewise epimorphisms}

   Fib = {surjections with fibrant kernel}.

   (injective model structure)

- C_c = all.
- C^+(injective) $\subset \mathcal{C}_f \subset C$(injectives)

- Fib $\cap$ W = surjections with injective kernel.

- X inj $\Leftrightarrow$ $\begin{cases} X \in \mathcal{C}_f \\ X \text{ acyclic.} \end{cases}$

Properties - $C(R)^{proj} \overset{id}{\longleftrightarrow} C(R)^{inj}$ Quillen eq.

- $f: R \to R'$ $\quad C(R)^{proj} \rightleftarrows C(R')^{proj}$

  $\exists$ Quillen adj.

$$X \longmapsto X \otimes_R R'$$

$$Y \longleftarrow\!\shortmid Y$$

Quillen eq $\Leftrightarrow$ f iso.

$$C(R)^{inj} \rightleftarrows C(R')^{inj}$$

Quillen adj $\Leftrightarrow$ R' flat over R.

- $M, N \in R\text{-Mod}$ $\quad [M[n], N] = \mathrm{Ext}^n(M,N)$.

<u>Th</u> $\mathcal{A}$ = Grothendieck ab. cat.

$\mathcal{C} = C.(\mathcal{A})$ $\quad W$ = q-iso. $\quad \exists$ model structures

1) $\mathrm{Cof} = \{\text{mono}\}$ $\quad$ (injective model structure)

2) $T \in \mathrm{Top}$ $\qquad \mathcal{A} = \mathrm{Sh}(T, \Lambda)$

Grothendieck top $\qquad\qquad \Lambda \in$ Rings.

$\mathcal{G}$ = generating family of T.

$$\mathcal{C}_f = \{K \mid \forall U \in \mathcal{G}, \forall n, \; H^n(\Gamma(U,K)) \simeq \mathcal{H}^n(U,K)\}$$

hypercohomology

$\mathrm{Fib} = \{ p : K \to L \mid$ - $\forall u \in \mathcal{G}$, $\Gamma(u, K) \xrightarrow{p_\#} \Gamma(u, L)$

degreewise surj with fibrant kernel $\}$

$\mathrm{Fib} \cap W = \{$ - ——— ——— + $\forall u \in \mathcal{G}$,

$\Gamma(u, \ker p)$ acyclic $\}$

s.t. $\forall u \in \mathcal{G}$, $\Lambda(u)$ is fibrant. (<u>projective model str.</u>)

<u>Simplicial sets</u> = combinatorical models for good topological spaces

<u>Def</u> $\Delta \subset \mathrm{FinSet}$

$\mathrm{Obj} = \{ [n] = \{0, 1, \dots, n\} \}$

$\mathrm{Hom}([n], [m]) = \{$non-decreasing maps$\}$

$x \geq y \Rightarrow f(x) \geq f(y)$

$0 \leq i \leq n$ $\quad$ $\underline{d^i} : [n-1] \to [n]$ $\quad$ "skip i"

$0 \leq i \leq n-1$ $\quad$ $\underline{s^i} : [n] \to [n-1]$ $\quad$ $\begin{matrix} i \\ i+1 \end{matrix} \rightrightarrows i$

<u>cosimplicial identities</u>

$$d^j d^i = d^i d^{j-1} \qquad i<j$$

$$s^j d^i = \begin{cases} d^i s^{j-1} & i<j \\ \mathrm{id} & i=j, j+1 \\ d^{i-1} s^j & i>j+1 \end{cases}$$

$$s^j s^i = s^{i-1} s^j \qquad i>j$$

- $\mathcal{C}$ category

$\mathrm{Fun}(\Delta, \mathcal{C})$ = <u>cosimplicial objects</u> in $\mathcal{C}$

$s\mathcal{C} = \mathrm{Fun}(\Delta^{op}, \mathcal{C})$ = <u>simplicial objects</u> in $\mathcal{C}$

$\mathcal{C}$ = Sets <u>sSets</u> = {simplicial sets}

$\mathcal{C}$ = Ab sAb = {simplicial abelian groups}

<u>Explicitly</u> $X \in$ sSets

$\Longleftrightarrow$ $X_n = X([n]) \in$ Sets $n \geq 0$ <u>set of n-simplices</u>

$d_i : X_n \to X_{n-1}$ $n \geq 1$ $0 \leq i \leq n$ <u>face maps</u>

$s_i : X_{n-1} \to X_n$ $n \geq 1$ $0 \leq i \leq n-1$ <u>degeneracy maps</u>

\+ <u>simplicial identities</u>

$$d_i d_j = d_{j-1} d_i \qquad i<j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & i<j \\ \mathrm{id} & i=j, j+1 \\ s_j d_{i-1} & i>j+1 \end{cases}$$

$$s_i s_j = s_j s_{i-1} \qquad i>j.$$

$$X = \left(\cdots \longrightarrow X_2 \; \begin{smallmatrix} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \\ \xleftarrow{s_1} \\ \xrightarrow{d_2} \end{smallmatrix} \; X_1 \; \begin{smallmatrix} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow[d_1]{} \end{smallmatrix} \; X_0 \right)$$

- $x \in X_n$ is non-degenerate if $\forall i$, $x \notin \mathrm{Im}(s_i)$

<u>Recall</u> Yoneda: $\mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$

$$a \longmapsto h_a = \mathrm{Hom}_{\mathcal{C}}(-, a)$$

fully faithful.

Def 1) $n \geq 0$ $\quad \Delta^n = h_{[n]} \in sSets$ <u>standard n-simplex</u>

- $(\Delta^n)_k = \mathrm{Hom}_{\Delta}([k], [n])$

- Δ^n has $\binom{n+1}{k+1}$ non-degenerate k-simplices

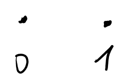
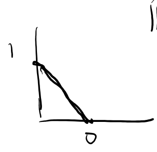
in part, 1 non-deg n-simplex.

- Yoneda $\Rightarrow$ $\forall X \in sSets$, $X_n = \mathrm{Hom}_{sSets}(\Delta^n, X)$

2) $E \subset [n] \Rightarrow \Delta^E \subset \Delta^{[n]}$ subfunctor.

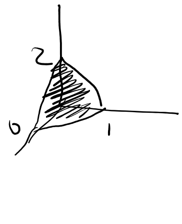
$$\partial\Delta^n = \bigcup_{E \subsetneq [n]} \Delta^E \subset \Delta^n \qquad \underline{\text{boundary}}$$

$n \geq 1$ $\quad 0 \leq k \leq n$ $\quad \Lambda^n_k = \bigcup_{k \in E \subsetneq [n]} \Delta^E \subset \Delta^n$ $\quad$ k-th <u>horn</u>

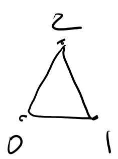
Ex $\Delta^1 =$  $\quad \partial\Delta^1 =$  

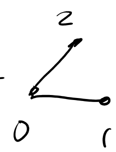
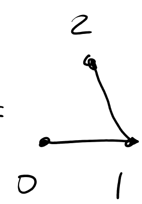
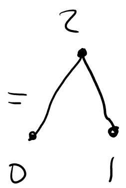
$\mathbb{R}^2$

$\Lambda^1_0 =$  $\quad \Lambda^1_1 =$ 



$\mathbb{R}^3$

$\Delta^2 =$  $\quad \partial\Delta^2 =$ 

$\Lambda^2_0 =$  $\quad \Lambda^2_1 =$  $\quad \Lambda^2_2 =$ 

$$(\Delta^1)_0 = \mathrm{Hom}([0], [1]) = \{0, 1\}$$

$(\Delta^1)_0 = \mathrm{Hom}([0],[1]) = \{0,1\}$

$$(\Delta^1)_1 = \mathrm{Hom}\left(\begin{smallmatrix}1\\ \vdots\\ 0\end{smallmatrix}[1], \begin{smallmatrix}1\\ \vdots\\ 0\end{smallmatrix}[1]\right) = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightrightarrows 0,\ \begin{pmatrix} 0\to 0 \\ 1 \to 1 \end{pmatrix} \checkmark \text{ non-deg.},\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightrightarrows 1 \right\}$$

$$(\Delta^n)^{nd}_K = \{ [k] \to [n] \mid \text{injective, non-deg.} \}.$$
$$= \binom{n+1}{k+1}$$

Geometric realizations

$$|\cdot| : sSets \longrightarrow Top \qquad \mathbb{R}^{n+1} \supset$$
$$\Delta^n \longmapsto |\Delta^n| = \{ (x_0, \dots x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0,\ \sum x_i = 1 \}.$$

For geneneral $X \in sSets$

Abstractly

Kan's Lemma $\mathcal{C}$ = cat with colimits

$A \in Cat$, $u : A \to \mathcal{C}$ functor

Then the evaluation functor $u^* : \mathcal{C} \to Fun(A^{op}, Sets)$

$$Y \longmapsto u^*Y : a \mapsto Hom_{\mathcal{C}}(u(a), Y)$$

has a left adjoint $u_! : Fun(A^{op}, Sets) \to \mathcal{C}$

s.t. $\forall a \in A$, $u(a) \simeq u_!(h_a)$

(left Kan extension)

D/ $X \in Fun(A^{op}, Sets)$ $\quad u_!(X) = \operatorname*{colim}_{\substack{h_a \to X \in Fun(A^{op}, Sets) \\ a \in A}} u(a)$ $\square$.

- $A = \Delta$, $\mathcal{C} = \mathrm{Top}$ $\quad u : \Delta \longrightarrow \mathrm{Top}$

$$[n] \longmapsto |\Delta^n|$$

$$\Rightarrow \quad \underset{\text{``}}{\overset{|\cdot|}{u_!}} : sSets \longrightarrow \mathrm{Top}$$

$$X \longmapsto |X| = \underset{\Delta^n \to X \in sSets}{\mathrm{colim}} |\Delta^n| \qquad \underline{\text{geometric realisation}}$$

left adjoint of $\overset{\mathrm{Sing}}{\overset{\text{``}}{u^*}} : \mathrm{Top} \longrightarrow sSets$

$$Y \longmapsto (\mathrm{Sing}\, Y)_n = \mathrm{Maps}(|\Delta^n|, Y)$$

$\underline{\text{singular simplicial set}}$

$$|\cdot| : sSets \rightleftarrows \mathrm{Top} : \mathrm{Sing}$$

__Explicitly__

$$|X| = \left(\coprod_{n \geq 0} X_n \times |\Delta^n| \right) \Big/ (X(f)(x), a) \sim (x, |\Delta^f|(a))$$

$$x \in X_n, \quad f : [m] \to [n]$$

$$a \in |\Delta^m|$$

with colimit top. induced by $(X_n \times |\Delta^n| \longrightarrow |X|)$

__Rk__ - $|X|$ is always a CW complex.

- $\{ n\text{-cells of } |X| \} \overset{bij}{\longleftrightarrow} \{ \text{non-deg } n\text{-simplices in } X_n \}$

__Model structure__

__Th__ $\exists$ model structure on sSets ($\underline{\text{Kan-Quillen model structure}}$)

structure)

s.t. – Cof = levelwise injections

– W = weak htp. equivalences

(i.e. $f: X \to Y$ s.t. $\pi_n(|f|)$ iso)

– Fib = $R\begin{pmatrix} \Lambda^n_k \\ \downarrow \\ \Delta^n \end{pmatrix}$ (Kan fibrations)

– Every object is cofibrant

– Fibrant objects are called Kan complexes

(∞-groupoids)

– $\mathrm{Fib} \cap W = R\begin{pmatrix} \partial\Delta^n \\ \downarrow \\ \Delta^n \end{pmatrix}$

Th $|\cdot| : \mathrm{sSets} \rightleftarrows \mathrm{Top} : \mathrm{Sing}$ is a Quillen equivalence.

(homotopy hypothesis)

Rk $\exists$ another model structure on sSets (Joyal model structure)

– Cof = mono.

– W = "weak categorical equivalences"

– Fib = "Isofibrations"

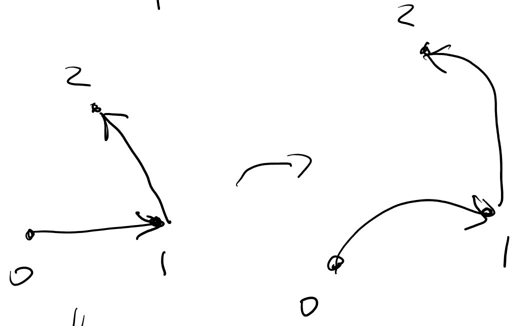
– Every object is cofibrant.

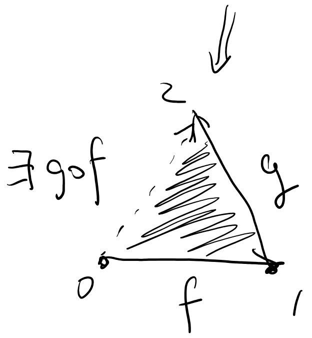
- Fibrant Objects = <u>quasi-categories</u>
  (<u>∞-categories</u>)

$$= \left\{ X \in sSets \;\middle|\; \begin{matrix} X \\ \downarrow \\ * \end{matrix} \in R\left(\begin{matrix} \Lambda^n_i \\ \downarrow \\ \Delta^n \end{matrix} \right), \ \forall\, 0<i<n \right\}$$

"all inner horns have fillers".

<u>Ex</u>:
$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^2 \arrow[r] \arrow[ur, dashed, "\exists"] & * \end{tikzcd}$$





map in X can be composed, but not uniquely.

- $f: A \to B$ <u>weak cat. eq</u>

$\iff$ $\forall X$ quasi-category

$$\tau(f^*): \tau(\underline{\mathrm{Hom}}(B,X)) \xrightarrow{\sim} \tau(\underline{\mathrm{Hom}}(A,X))$$

$$i: \Delta \to Cat \xrightarrow{\text{Kan's lemma}} N = i^*: Cat \to sSets$$

$$\mathcal{C} \longmapsto N(\mathcal{C})$$

$$N(\mathcal{C})_n = \underset{x_0}{\bullet} \to \underset{x_1}{\bullet} \to \bullet \to \cdots \to \underset{x_n}{\bullet} \qquad [n] \longmapsto \underline{\mathrm{Hom}_{Cat}([n], \mathcal{C})}$$

<u>nerve</u> of a category

$\tau = i_! : sSets \to Cat$ <u>fundamental category</u> of a simplicial set

<u>Rk</u> $X \in sSets$ is the nerve of a category

$\Leftrightarrow$ "all inner horns have <u>unique</u> fillers"