Maximal real varieties via moduli spaces

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Real geometry

Alg. geom A real alg. var is a variety $X_0/\mathbb{R}$

$\longrightarrow$ complex var $X = X_0 \times_{\mathbb{R}} \mathbb{C}$ $\rightsquigarrow$ $X(\mathbb{C}) = X_0(\mathbb{C})$ (analytic space)

- $X_0(\mathbb{R})$ subspace in $X(\mathbb{C})$ $\quad X(\mathbb{C})^\iota = X_0(\mathbb{R})$
- involution $\iota \circlearrowright X(\mathbb{C})$ induced by complex conjugation on $\mathbb{C}$ (not algebraic)

Complex geom A real var $=$ (X, ι) complex variety X + anti-holomorphic involution ι $\quad(\iota^* I = -I)$ (real structure)

Rk 2 pts of view are equivalent

$(X, \iota) \longmapsto X/\iota$

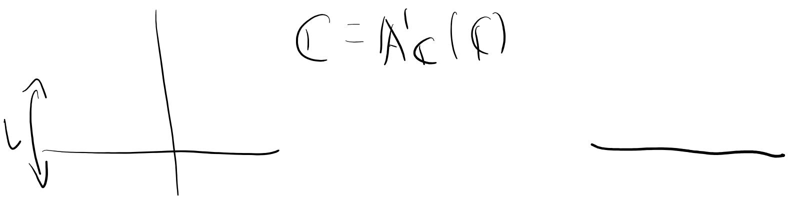
$(\mathrm{Spec}\, A, \iota) \longmapsto \mathrm{Spec}\, A^\iota$

Ex 1 $X_0 = \mathbb{A}^1_{\mathbb{R}} = \mathrm{Spec}\, \mathbb{R}[T]$

$X = \mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\, \mathbb{C}[T]$

$\iota \circlearrowright X$ induced by $\mathbb{C}[T] \longrightarrow \mathbb{C}[T]$

$\mathbb{C} = \mathbb{A}^1_{\mathbb{C}}(\mathbb{C})$



$X_0(\mathbb{R})$

<u>Ex 2</u> $X = \mathbb{CP}^1 = \operatorname{Proj}(\mathbb{C}[T_0, T_1])$

$\iota : X \longrightarrow X$

$[x_0 : x_1] \mapsto [\bar{x}_0 : \bar{x}_1]$ Standard real structure

$X(\mathbb{R}) := X^{\iota} = \{ [x_0 : x_1] \mid x_0, x_1 \in \mathbb{R} \} = \mathbb{RP}^1$

$X_0 = \operatorname{Proj}(\mathbb{R}[T_0, T_1]) = \mathbb{P}^1_{\mathbb{R}}$

<u>Ex 3</u> $Y = \mathbb{CP}^1$ $\iota : Y \longrightarrow Y$

$[x_0 : x_1] \longmapsto [-\bar{x}_1 : \bar{x}_0]$ anti-holo inv.

$z \longmapsto -\frac{1}{\bar{z}}$

$Y(\mathbb{R}) = Y^{\iota} = \emptyset$

$Y_0 =$ real conic without real points

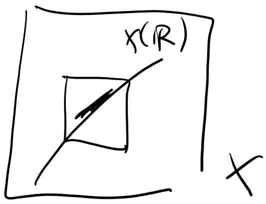
$\simeq \operatorname{Proj}(\mathbb{R}[u, v, w] / u^2 + v^2 + w^2)$

<u>Theme</u> Study the topology of $X(\mathbb{R})$ in terms of complex geometry of X

<u>Rk</u> (X, ι) smooth real var. $\dim = n$

Then $X(\mathbb{R}) \subset X(\mathbb{C})$ submfd $\dim_{\mathbb{R}} = n$

if $X(\mathbb{R}) \neq \emptyset$



<u>Th 1</u> (Smith–Thom inequality)

(X, ι) real variety

total $\mathbb{F}_2$ - Betti number

$$\sum_i b_i(X(\mathbb{R}), \mathbb{F}_2) \leq \sum_j b_j(X, \mathbb{F}_2)$$

<u>Rk</u> – Th 1 comes from Smith theory

– Key point: $\iota \curvearrowright C_\cdot(X) = \mathbb{F}_2$ singular chain complex

SES

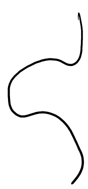
$$0 \to C_\cdot(X^\iota) \oplus (1+\iota)C_\cdot(X) \longrightarrow C_\cdot(X) \xrightarrow{1+\iota} (1+\iota)C_\cdot(X) \to 0$$

<u>Def</u> A real variety (X, ι) is called <u>maximal</u> (M-variety) if "=" holds in Thm 1.

<u>Ex</u> – $(\mathbb{CP}^n$, standard$)$, Grassmanian, Flag var.

– Elliptic curve

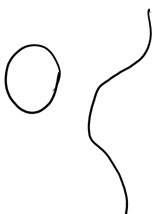
$E(\mathbb{R}) =$



S^1

$b = 2$

or



$S^1 \sqcup S^1$

$b = 4$

$b=2$ $\qquad\qquad$ $b=4$

not maximal $\qquad\qquad$ maximal

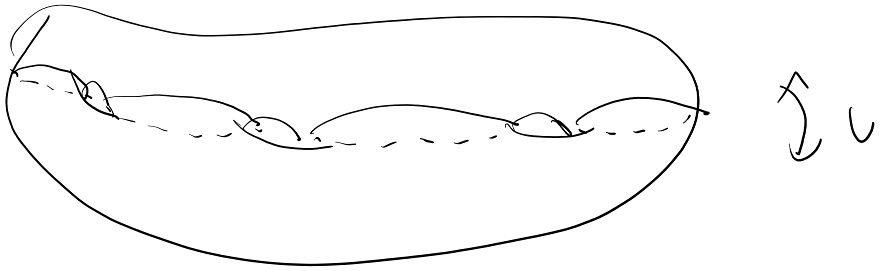
Th 1 in case of dim 1 (special case)

<u>Th</u> (Harnack - Klein)

X compact Riemann surface genus $= g$

ι real str.

then $X(\mathbb{R}) = \coprod_{i=1}^{N} S^1$ with $N \leq g+1$



<u>Constructions of max. real var.</u>

- $\mathbb{P}^n$, Grassmannian, flag var., toric var.

  (w./ standard real str.)

- product, disjoint union

- projective bundle $\quad \mathbb{P}(E) \to X$

- blowup $\quad \mathrm{Bl}_Y X \quad$ X maximal

  Y maximal, inv. under ι

. - -

<u>Prop</u> (X,ν) real var. TFAE $\qquad G = \mathbb{Z}/2 \curvearrowright X$

1) (X,ν) is maximal

2) $H^*_G(X;\mathbb{F}_2) \twoheadrightarrow H^*(X,\mathbb{F}_2)$ surjective

3) $\begin{cases} G \text{ acts trivially on } H^*(X;\mathbb{F}_2) \\ \text{Leray-Serre ss} \quad E_2^{p,q} = H^p(G, H^q(X,\mathbb{F}_2)) \Rightarrow H^{p+q}_G(X,\mathbb{F}_2) \\ \text{degenerates} \end{cases}$

<u>Rk</u> - $G \curvearrowright EG \ (= S^\infty)$
$$\downarrow$$
$$BG \ (= \mathbb{R}\mathbb{P}^\infty)$$

$X \hookrightarrow X \times EG/G = X_G$
$$\downarrow$$
$$BG$$

$H^*_G(X;\mathbb{F}_2) = H^*(X \times EG/G, \mathbb{F}_2)$ Borel

<u>Def</u> $2) \Leftrightarrow G \curvearrowright X$ is equivariantly formal

<u>Def</u> Equivariantly formal "motive"

$M \in Mot_{hom}(\mathbb{R})$ is <u>equi. formal</u> if

$\shortparallel$

(X,p,n) $\qquad H^*_G(M,\mathbb{F}_2) \twoheadrightarrow H^*(M;\mathbb{F}_2)$

where both sides are defined by realization

$$Mot_{hom}(\mathbb{R}) \longrightarrow \mathbb{F}_2\text{-v.s.}$$

$$X \longmapsto H^*(X(\mathbb{C}); \mathbb{F}_2)$$

Prop 1) M_1, M_2 EF $\Rightarrow$ $M_1 \otimes M_2$ EF

2) $M = M_1 \oplus M_2$ then M EF $\Leftrightarrow$ M_1, M_2 EF

3) $X_1, \dots, X_n$ max. real var

Y motivated by X_i (i.e. $M(Y) \in \langle M(X_i) \rangle^{\otimes}$)

$\Rightarrow$ Y maximal

☆ patch working (Viro, Itenberg)

New constructions

Results Th 1 (Brugallé – Schaffhausen)

$C/\mathbb{R}$ max curve $(n,d)=1$ $n>0$

$M = M_C(n,d) = \{$stable VB on C, $rk = n$, $deg = d\}$

is maximal real var.

Pf/ compute all Betti numbers (Atiyah-Bott, Zagier ~) $\square$.

A quick proof : $\Rightarrow H^*(M, \mathbb{F}_2)$

1) A-B $\Rightarrow$ $H^*(M, \mathbb{Z})$ is torsion free ($\mathbb{F}_2$-alg) generated as alg by

U (i) Künneth ($c_i(U)$ ...

$$\begin{array}{c} \mathcal{U} \\ \downarrow \text{ univ. VB} \\ M \times C \xrightarrow{\pi} M \end{array}$$

(i) Künneth component of $c_j(\mathcal{U})$ $\quad j = 1, \dots, n$

(ii) $c_j\big(R\pi_*(\mathcal{U})\big)$

2) To show $H^*_G(M, \mathbb{F}_2) \twoheadrightarrow H^*(M, \mathbb{F}_2)$,

Suffices: (i) & (ii) are in the image

For (i) $\quad c_j^G(\mathcal{U})(\tilde{\alpha}) \longmapsto c_j(\mathcal{U})_*(\alpha) \in H^*(M, \mathbb{F}_2)$

$H^*_G(C \times M, \mathbb{F}_2) \ni c_j^G(\mathcal{U}), \quad \tilde{\alpha} \in H^*_G(C, \mathbb{F}_2) \twoheadrightarrow H^*(C, \mathbb{F}_2) \ni \alpha$

For (ii) $\quad c_j^G(R\pi_*(\mathcal{U})) \longmapsto c_j(R\pi_*(\mathcal{U}))$

<u>Other results</u> - stable Higgs bundle

Markman

- stable sheaves on surfaces

($\mathbb{P}^2$, Poisson surface)

- Hilbert schemes on surfaces

Rohklin congruence