Globalization and coherent duality

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§1 Grothendieck's coherent duality

S scheme $\qquad D_{Qcoh}(S) \underset{\text{full}}{\subset} D(\mathcal{O}_S)$

Q-coh cohomology

<u>Th</u> (Grothendieck)

$f : X \to Y \in Sch_k^{ft}$ $\qquad k$ field

separated

1) f proper $Rf_* : D_{Qcoh}(X) \to D_{Qcoh}(Y)$ has a right adjoint

$$f^! : D_{Qcoh}(Y) \to D_{Qcoh}(X)$$

Moreover, $\forall \mathcal{F} \in D_{Qcoh}(X)$, $\mathcal{G} \in D^b_{Qcoh}(Y)$

counit $Rf_* f^! \mathcal{G} \to \mathcal{G}$ induces

$$Rf_* R\underline{Hom}_X(\mathcal{F}, f^!\mathcal{G}) \simeq R\underline{Hom}_Y(Rf_*\mathcal{F}, \mathcal{G})$$

2) f finite Tor dimension

$$f^!\mathcal{G} \simeq f^!\mathcal{O}_Y \otimes^L_{\mathcal{O}_Y} f^*\mathcal{G}.$$

3) f smooth, $f^!\mathcal{O}_Y \simeq \Omega^n_{X/Y}[n]$ $\qquad n = \dim f$

Ex (Serre duality) $X = \mathbb{P}^n_k \quad Y = \mathrm{Spec}\, k$

$$f^! \mathcal{O}_Y \simeq \omega_{\mathbb{P}^n}[n] \simeq \mathcal{O}_{\mathbb{P}^n_k}(-n-1)\,[n]$$

$\Rightarrow R\mathcal{H}om_k(\mathcal{F}, \omega_{\mathbb{P}^n}[n]) \xrightarrow{\sim} R\mathcal{H}om_k(R\Gamma(\mathbb{P}^n, \mathcal{F}), k)$

$\xrightarrow{H^{i-n}} \mathrm{Ext}^i(\mathcal{F}, \omega_{\mathbb{P}^n}) \xrightarrow{\sim} H^{n-i}(\mathbb{P}^n, \mathcal{F})^\vee$ as k-vector spaces

§2 Classical approaches

Hartshorne explicit study on dualizing complexes

X/k FT, $C \in D^+_{coh}(X)$

C is a dualizing object if

i) C has finite inj. dimension

ii) $\mathcal{O}_X \xrightarrow{\sim} R\mathcal{H}om_X(C, C)$

Facts $f: X \to Y$ $\quad C$ dualizing object on Y

1) f finite, $f^b C = \bar{f}^* R\mathcal{H}om(f_* \mathcal{O}_X, C) \in D_{coh}(X)$

is a dualizing object $\qquad \bar{f}^* : \mathrm{Mod}(f_* \mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X)$

$$(X, \mathcal{O}_X) \to (Y, f_* \mathcal{O}_X)$$

2) f smooth rel. dim d

$$f^\# C = f^* C \otimes \omega_{X/Y}[d] \quad \text{is a dualizing object}$$

3) dualizing object unique up to $\otimes$-invertible object and shift

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4) dualizing object exists in many cases

$f: X \to Y \in Sch_k^{FT}$ $\qquad$ C duality on Y

$$f^! : D_{coh}(Y) \to D_{coh}(X)$$
$$F \longmapsto Lf^* F \otimes_{\mathcal{O}_X} f^! C$$

<u>Neeman</u> <u>Brown representability</u>

<u>Th</u> T compactly gen. Δ-cat.

$F: T^{op} \to Ab$ | exact, commutes with coproducts

Then F is representable

<u>Prop</u> $f: X \to Y \in Sch_k^{FT}$ $\quad L \in D_{Qcoh}(Y)$

Then $D_{Qcoh}(X)^{op} \longrightarrow Ab$

$$K \longmapsto Hom_{D_{Qcoh}(Y)}(Rf_* K, L)$$

is rep. In part, Rf_* has a right adjoint

<u>Deligne</u> Ind / pro objects

Verdier: $X \xrightarrow{f} Y$ proper, Rf_* has a right adjoint $f^!$

For general f

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ & Y \end{tikzcd}$$

$\bar{f}$ proper

$\mathcal{I}$ = ideal sheaf $\bar{X} \setminus X \hookrightarrow \bar{X}$

$$\mathrm{Coh}(X) \longrightarrow \mathrm{pro}\,\mathrm{Coh}(\bar{X}) \qquad \tilde{\mathcal{F}} \in \mathrm{Coh}(\bar{X})$$

$$\mathcal{F} \longmapsto \text{"}\lim\text{"}\, \mathcal{I}^n \tilde{\mathcal{F}} \qquad \text{s.t. } j^* \tilde{\mathcal{F}} = \mathcal{F}$$

$$\mathrm{Hom}(\lim -, \lim -) = \mathrm{colim}\,\lim\, \mathrm{Hom}(-,-)$$

$$\rightsquigarrow j_! : \mathrm{pro}\, D^b_{coh}(X) \longrightarrow \mathrm{pro}\, D^b_{coh}(\bar{X})$$

<u>Facts</u> $\mathcal{G} \in \mathrm{Qcoh}(\bar{X})$

$$\mathrm{Hom}_X(\mathcal{F}, j^*\mathcal{G}) = \mathrm{Hom}_{\bar{X}}(\text{"}\lim\text{"}\, \mathcal{I}^n \tilde{\mathcal{F}}, \mathcal{G})$$

$$\Vert \qquad\qquad\qquad \Vert$$

$\mathcal{G} = \text{"}\mathrm{colim}\text{"}\, \mathcal{G}_i$

$\mathrm{Ind}\,\mathrm{Coh}(\bar{X}) = \mathrm{Qcoh}(\bar{X})$

$\mathcal{G}_i \in \mathrm{Coh}(\bar{X})$

$$\mathrm{colim}\, \mathrm{Hom}_X(\mathcal{F}, j^*\mathcal{G}_i) = \mathrm{colim}\, \mathrm{colim}\, \mathrm{Hom}_{\bar{X}}(\mathcal{I}^n \tilde{\mathcal{F}}, \mathcal{G}_j)$$

We define $f_! : \mathrm{pro}\, D^b_{coh}(X) \longrightarrow \mathrm{pro}\, D^b_{coh}(Y)$

We define

$$\begin{tikzcd} f_! : \mathrm{pro}\, D^b_{coh}(X) \arrow[r] \arrow[d, "j_!"'] & \mathrm{pro}\, D^b_{coh}(Y) \\ \mathrm{pro}\, D^b_{coh}(\bar{X}) \arrow[ur, "Rf_*"'] & \end{tikzcd}$$

§3 Clausen–Scholze : condensed math

Def A discrete adic space = triple $(X, \mathcal{O}_X, (|\cdot|_x)_{x \in X})$

$X \in \mathrm{Top}$, $\mathcal{O}_X$ sheaf of rings on X

$|\cdot|_x$ = eq. class of valuation on x

Locally of the form $\left(\mathrm{Spa}(A, A^+), \mathcal{O}_{\mathrm{Spa}(A,A^+)}, (|\cdot|_x)_{x \in \mathrm{Spa}(A,A^+)} \right)$

(affinoid adic space)

For $R \to A \in \mathrm{Alg}_{\mathbb{Z}}^{fg} \rightsquigarrow$ affinoid adic space $\mathrm{Spa}(A, \tilde{R})$

$\tilde{R}$ = integral closure of R in A

$\rightsquigarrow \mathrm{Mod}^{\mathrm{cond}}_{(A,R)_\blacksquare}$ solid $(A,R)_\blacksquare$-modules

$\rightsquigarrow D((A,R)_\blacksquare) = D(\mathrm{Mod}^{\mathrm{cond}}_{(A,R)_\blacksquare})$

Th (Globalization) X discrete adic space

1) $U = \mathrm{Spa}(A, A^+) \longmapsto D((A, A^+)_\blacksquare)$

defines a sheaf of ∞-cats on X, denoted by $D(X_\blacksquare)$

2) $f : X \to Y \in Sch^{ft}_k$

$Z := X^{ad/Y}$ (locally $f : \mathrm{Spec}\, \overset{\mathcal{U}}{\overset{\shortparallel}{A}} \to \mathrm{Spec}\, \overset{V}{\overset{\shortparallel}{R}}$, Z is given by $\mathrm{Spa}(A, \hat{R})$)

$X^{ad} = X^{ad/X}$

f induces $X^{ad} \overset{j}{\hookrightarrow} Z = X^{ad/Y} \xrightarrow{g} Y^{ad}$

open imm

if f sep

$$\mathcal{D}(X^{ad}_{\blacksquare}) \underset{j^*}{\overset{j_*}{\rightleftarrows}} \mathcal{D}(X^{ad/Y}_{\blacksquare}) \underset{g^*}{\overset{g_*}{\rightleftarrows}} \mathcal{D}(Y^{ad}_{\blacksquare})$$

<u>Th</u> f sep in $Sch^{FT}_{\mathbb{Z}}$

1) j^* has a left adj $j_!$

2) $f_! = g_* \circ j_! : \mathcal{D}(X^{ad}_{\blacksquare}) \to \mathcal{D}(Y^{ad}_{\blacksquare})$

3) $f_!$ has a right adj $f^!$

$f^!$ preserves discrete objects

4) g finite Tor-dimension $\Rightarrow$ $f^!$ preserve compacts

$$f^! M = f^* M \otimes^L_{(\mathcal{O}_{X^{ad}}, \mathcal{O}^+_{X^{ad}})_{\blacksquare}} f^!(\mathcal{O}_{Y^{ad}}, \mathcal{O}^+_{Y^{ad}})_{\blacksquare}$$

<u>Cor</u> (coherent duality)

$f: X \to Y \in Sch^{ft}_{\mathbb{Z}}$ proper

Then Rf_* has a right adjoint $f^!: D(Y^{ad}_{\blacksquare}) \longrightarrow D(X^{ad}_{\blacksquare})$

on the subcat of discrete obj for $f: X^{ad} \longrightarrow Y^{ad}$