- condensed $\begin{matrix} X \\ \text{set} \\ \cancel{\text{gp}} \\ \text{ab} \\ \text{rng} \end{matrix}$ = sheaf of $\cancel{\emptyset}$ X on $*$ proet

= $\{\text{profinite sets}\}^{op} \longrightarrow$ Sets

1) commute with finite ~~co~~products

2) $\forall\, S' \twoheadrightarrow S$

$F(S) \simeq \{ x \in F(S') \mid p_1^* x = p_2^* x \in F(S' \times_S S') \}$

<u>Ex 1.5</u> $\mathrm{Top} \longrightarrow \mathrm{Cond}(\mathrm{Set})$

$X \longmapsto \underline{X} : S \mapsto C^0(S, X)$

S = compact Hausdorff space

- <u>extremally disconnected</u> if $\forall S' \in$ CHaus. $\forall\, S' \twoheadrightarrow S$ splits

<u>Prop. 2.7</u> $\mathrm{Sh}(\mathrm{EDS}) \xrightarrow{\sim} \mathrm{Sh}(*\text{pro-et})$

$\wr\!\downarrow$ eq. of categories

$\mathrm{PSh}^{\amalg}(\mathrm{EDS})$

<u>pre-analytic ring</u> A = - $\underline{A}$ condensed ring, $\mathrm{Mod}^{\mathrm{cond}}_{\underline{A}}$ = $\underline{A}$-modules in Cond(Ab)

- functor $\{\mathrm{EDS}\} \longrightarrow \mathrm{Mod}^{\mathrm{cond}}_{\underline{A}}$ commuting with finite ~~co~~products

$S \longmapsto A[S]$

- natural tr. $S \longrightarrow A[S]$

~~Ex $A \in \mathrm{Rng}$ $(A, \mathbb{Z})_{\blacksquare} = (A, S \mapsto \mathbb{Z}_{\blacksquare}[S] \otimes A)$~~

<u>Rk</u> $\forall M \in \mathrm{Mod}_R$, $\forall I$, $M \otimes^L_R \prod_I R \simeq M \otimes_R \prod_I R \simeq \prod_I M$. ~~MMA~~ M fin.gen., then use res. by fin. free R-Mod.

- A is <u>analytic</u> if $\forall$ $C = (\cdots \to C_i \to \cdots \to C_1 \to C_0 \to 0) \in \mathrm{Cpl}(\mathrm{Mod}^{\mathrm{cond}}_{\underline{A}})$

s.t. $\forall C_i = \bigoplus A[T_{ij}]$ $T_{ij} \in \mathrm{EDS}$

$\forall S \in \mathrm{EDS}$ $\underline{R\mathrm{Hom}}_{\underline{A}}(A[S], C) \xrightarrow{\sim} \underline{R\mathrm{Hom}}_{\underline{A}}(\underline{A}[S], C] \in \mathrm{Cond}(\mathrm{Ab})$

<u>Ex</u> A = f.g. $\mathbb{Z}$-alg. $A_{\blacksquare} = (A, (S = \lim S_i) \longmapsto (A_{\blacksquare}[S] = \lim A[S_i]))$

$R \to A$ f.g. $\mathbb{Z}$-alg $(A, R)_{\blacksquare} = (A, (A, R)_{\blacksquare}[S] = R_{\blacksquare}[S] \otimes_R A)$

$\underline{A}$ = analytic ring

$$\mathrm{Mod}_A^{\mathrm{cond}} \subset \mathrm{Mod}_{\underline{A}}^{\mathrm{cond}}$$

$$\| $$

$$\{ M \in \mathrm{Mod}_A^{\mathrm{cond}} \mid \forall S \in \mathrm{EDS},\ \mathrm{Hom}_{\underline{A}}(\underline{A}[S], M) \xrightarrow{\sim} M(S) \}.$$

$$D(A) := D(\mathrm{Mod}_A^{\mathrm{cond}})$$

Th. 8.13

I) $R \to A \in \mathrm{Alg}^{fg}_{/\mathbb{Z}}$ ~~f.g. $\mathbb{Z}$-algebras~~

The pre-analytic ring $(A,R)_\blacksquare$ is analytic.

II) $R \to S \to A \in \mathrm{Alg}^{fg}_{/\mathbb{Z}}$

The forgetful functor $j_* : D((A,S)_\blacksquare) \hookrightarrow D((A,R)_\blacksquare)$

has a left adjoint $j^* : (-) \otimes^L_{(A,R)_\blacksquare} (A,S)_\blacksquare$

And j^* has a left adjoint $j_! : D((A,S)_\blacksquare) \to D((A,R)_\blacksquare)$

s.t. $j_! j^*(-) = (-) \otimes_{(A,R)_\blacksquare} j_! A$.

III). $f : \mathrm{Spec} A \to \mathrm{Spec} R \in \mathrm{Sch}^{ft}_{\mathbb{Z}}$

Define $f_! : D(A_\blacksquare) \xrightarrow{j_!} D((A,R)_\blacksquare) \to D(R_\blacksquare)$

Then - $f_!$ commutes with $\oplus$

- $\forall M \in D(R_\blacksquare)$, $N \in D(A_\blacksquare)$,

$$f_!\left((M \otimes^L_{R_\blacksquare} A_\blacksquare) \otimes^L_{A_\blacksquare} N \right) \simeq M \otimes^L_{R_\blacksquare} f_! N$$

- f finite Tor-dimension $\Rightarrow$ $f_!$ preserves compact objects

- $(g \circ f)_! \simeq g_! \circ f_!$

~~IV~~ 4) f as in 3), • $f_!$ has a right adjoint

$$f^! : D(R_\blacksquare) \to D(A_\blacksquare)$$

s.t. – $f^! R \in D(A_\blacksquare)$ is discrete, left bounded & f.g.
(relative dualizing complex)

– f finite Tor-dimension $\Rightarrow$ $f^! R$ bounded

$f^!$ commutes with direct sums

$f^!(-) = (- \otimes^L_{R_\blacksquare} A_\blacksquare) \otimes^L_{A_\blacksquare} f^! R$.

– f complete intersection $\Rightarrow$ $f^! R$ is invertible

– $(g \circ f)^! \simeq f^! \circ g^!$.

– $\forall M \in \mathrm{Mod}_R$, $\forall I$, $M \otimes^L_R \prod_I R \simeq M \otimes_R \prod_I R$; WMA M fin. gen.
LHS $= \prod_I M$ (use res. by fin. free R-Mod)

Rk – $f^!$ induces $D(R) \to D(A)$ (usual one)

– $f_! = Rf_*$ for f proper

in general $f_!$ doesn't preserve discrete objects (doesn't exist classically)

Ex $f : \mathrm{Spec}\, \underset{\mathbb{Z}[X,Y]/XY}{\overset{A}{\shortparallel}} \to \mathrm{Spec}\, \mathbb{Z}$

$A_\infty := \mathbb{Z}((X^{-1})) \times \mathbb{Z}((Y^{-1}))$ "functions near the boundary"

– $f_! A[1] = A_\infty / A = \big(\mathbb{Z}((X^{-1})) \times \mathbb{Z}((Y^{-1}))\big) \big/ (\mathbb{Z}[X,Y]/XY)$

– $f^! \mathbb{Z} = R\underline{\mathrm{Hom}}(A_\infty / A, \mathbb{Z})[1]$. dualizing complex.

Now assume $A = \mathbb{Z}[T]$ $R = \mathbb{Z}$ $A_\infty = \mathbb{Z}((T^{-1}))$ "functions near the boundary"
$\in \mathrm{Cond}(\mathrm{Alg}_A)$

$\Rightarrow A_\infty \in D((A,\mathbb{Z})_\blacksquare)$

Properties 1) $\exists$ ses $0 \to \mathbb{Z}[[U]] \otimes_{\mathbb{Z}} A \xrightarrow{\times(UT-1)} \overbrace{\mathbb{Z}[[U]] \otimes_{\mathbb{Z}} A}^{\shortparallel \prod_I \mathbb{Z}} \to A_\infty \to 0$
$\underbrace{\qquad\qquad\qquad\qquad}_{\text{compact proj in } \mathrm{Mod}^{\mathrm{Cond}}_{(A,\mathbb{Z})_\blacksquare}}$ $\quad \cancel{U} \mapsto T^{-1}$

$\Rightarrow A_\infty \in D((A,\mathbb{Z})_\blacksquare)$ compact

2) $A_\infty \in D((A,\mathbb{Z})_\blacksquare)$ idempotent

i.e. $A_\infty \otimes^L_{(A,\mathbb{Z})_\blacksquare} A_\infty \xrightarrow{\sim} A_\infty$

$\Rightarrow$ - $\{A_\infty\text{-modules}\} \subset D((A,\mathbb{Z})_\blacksquare)$ full subcat.

- $\forall M \in D((A,\mathbb{Z})_\blacksquare)$ has at most one A_∞-module structure

it exists $\Leftrightarrow$ $M \xrightarrow{\sim} M \otimes^L_{(A,\mathbb{Z})_\blacksquare} A_\infty$

3). For $C \in D(\mathrm{Cond}(A))$ with each term $\bigoplus \prod A$

$R\underline{\mathrm{Hom}}_A(A_\infty, C) = 0$.

Pf/ $(A,\mathbb{Z})_\blacksquare$ condensed $\Rightarrow D((A,\mathbb{Z})_\blacksquare) \subset D(\mathrm{Cond}(A))$ stable under lim colim

$\Rightarrow C \in D(A,\mathbb{Z})_\blacksquare)$

$C = \lim C^{\leq n} \Rightarrow$ WMA C connective

$A_\infty \in D(A,\mathbb{Z})_\blacksquare)$ compact $\Rightarrow$ WMA $C = \prod_I A$, even $C = A$

Suffices: $R\underline{\mathrm{Hom}}_A(A_\infty, A) = 0$ $\quad A[u^{-1}]/A \xrightarrow{uT-1} A[u^{-1}]/A$

$= \mathrm{Ker}\left(R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[[u]], A) \xrightarrow{uT-1} R\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathbb{Z}[[u]], A)\right) = 0$ $\square$.

4). $\forall I$, $\mathrm{coker}\left(A \otimes_{\mathbb{Z}} \prod_I \mathbb{Z} \hookrightarrow \prod_I A\right)$ is A_∞-module.

Pf/ $\mathrm{coker}(-) = \mathrm{coker}\left(\mathbb{Z}((T^{-1})) \otimes_{\mathbb{Z}[[T^{-1}]]} \prod_I \mathbb{Z}[[T^{-1}]] \longrightarrow \prod_I \mathbb{Z}((T^{-1}))\right)$

$\in \mathrm{Mod}(\mathbb{Z}((T^{-1})))$ $\square$.

5) $\mathbb{Z}[T]_\blacksquare$ is analytic

~~Pf of 8.13. I for $A = \mathbb{Z}[T]$~~

Pf/ C = good $\quad$ S profinite $\quad$ Need: $R\underline{\mathrm{Hom}}_A(A[S], C) \simeq R\underline{\mathrm{Hom}}_A(A_\blacksquare[S], C)$

Have: $C \in D((A,\mathbb{Z})_\blacksquare) \Rightarrow R\underline{\mathrm{Hom}}_A(A[S], C) \simeq R\underline{\mathrm{Hom}}_A((A,\mathbb{Z})_\blacksquare[S], C)$

$\mathbb{Z}_\blacksquare[S] = \prod_I \mathbb{Z} \Rightarrow$ Suffices: $R\underline{\mathrm{Hom}}_A(A \otimes_{\mathbb{Z}} \prod_I \mathbb{Z}, C) \simeq R\underline{\mathrm{Hom}}_A(\prod A, C)$

Follows from 3) & 4) $\square$.

Cor $j_* : D(A_\blacksquare) \to D((A,\mathbb{Z})_\blacksquare)$ full inclusion

$\exists$ left adj $j^* = (-) \otimes^L_{(A,\mathbb{Z})_\blacksquare} A_\blacksquare : D((A,\mathbb{Z})_\blacksquare) \to D(A_\blacksquare)$

Prop 6) $\operatorname{Ker}(j^*) = \{ A_\infty\text{-modules} \}$.

(follows from 2) 3) 4))

7) $j_! : D((A,\mathbb{Z})_\blacksquare) \to D((A,\mathbb{Z})_\blacksquare)$

$M \mapsto M \otimes^L_{A \otimes_\mathbb{Z} \mathbb{Z}_\blacksquare} (A_\infty/A)[-1]$

factors ~~uniquely~~ as $j_! : D(A_\blacksquare) \to D((A,\mathbb{Z})_\blacksquare)$

- fully faithful
- left adj. to j^*

D/ use: $\forall M, N \in D((A,\mathbb{Z})_\blacksquare)$

$\underline{RHom}_A(j_! M, N) \overset{2)+6)}{\simeq} \underline{RHom}_A(j_! M, N \otimes^L_{(A,\mathbb{Z})_\blacksquare} A_\blacksquare) \overset{(3)}{\xleftarrow{\sim}} \underline{RHom}_A(M, N \otimes^L_{(A,\mathbb{Z})_\blacksquare} A_\blacksquare)$. D.

~~Rk This proves 8.13. II~~ proves II

Rk Similar to $\mathbb{Z}_\blacksquare$, $R \mapsto \mathbb{Z}((T^{-1}))$.

8) $\forall I$, $j_! \prod_I A \xrightarrow{\sim} \prod_I (A_\infty/A)[-1]$

$j_! \prod_I A \simeq j_! j^* (A \otimes_\mathbb{Z} \prod_I \mathbb{Z}) = (A \otimes_\mathbb{Z} \prod_I \mathbb{Z}) \otimes^L_{(A,\mathbb{Z})_\blacksquare} (A_\infty/A)[-1]$

$= \prod_I \mathbb{Z} \otimes^L_{\mathbb{Z}_\blacksquare} (A_\infty/A)[-1] = \prod_I (A_\infty/A)[-1]$

For IV: - $f_!$ commutes with $\oplus$ $\Rightarrow$ has right adjoint $f^!$

- $f_!$ preserve compacts $\Leftrightarrow$ $f^!$ commutes with $\oplus$ (use 8))

- $f^! \mathbb{Z} = \underline{RHom}_\mathbb{Z}(\underbrace{f_! A}_{\text{compact}}, \mathbb{Z}) = \underline{RHom}_\mathbb{Z}(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T], \mathbb{Z})[1]$

$\Rightarrow f^! \mathbb{Z}$ discrete $\qquad \simeq \mathbb{Z}[T][1]$

General case

I: $R_\blacksquare[S] \otimes^L_R A \simeq R_\blacksquare[S] \otimes_R A$ $\Rightarrow$ reduce $(A,R)_\blacksquare$ to $R_\blacksquare$, or $A_\blacksquare$ for $A \in \mathrm{Alg}^{fg}_{\mathbb{Z}}$

choose $R = \mathbb{Z}[X_1, \dots, X_n] \twoheadrightarrow A$ $\rightsquigarrow$ reduce to $(A,R)_\blacksquare$ then $R_\blacksquare$

$\rightsquigarrow$ use induction $R[T]/R$.

II. WMA $S \twoheadrightarrow A$ ~~then reduce~~ $S = R[X_1, \dots, X_n]$, then induction

III, IV: similar: affine morphism reduces to closed immersion & $A[T]$. $\square$