Analytic rings

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$$\mathrm{Solid} \subset \mathrm{Cond}(\mathrm{Ab})$$

solidification $(-)^{\blacksquare}$

$$\forall\, S = \lim_i S_i \in \mathrm{Prof} \qquad \mathbb{Z}[S]^{\blacksquare} := \lim \mathbb{Z}[S_i]$$

$$\mathrm{Solid} = \{ C \in \mathrm{Cond}(\mathrm{Ab}) \mid \mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, C) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}[S], C) \}$$

Condensed rings : ring = unital assoc. ring

<u>Def</u> A <u>pre-analytic ring</u> $\mathcal{A}$ is a triple

$$\mathcal{A} = (\underline{\mathcal{A}}, \mathcal{A}[-], \tau) \quad \text{s.t.}$$

- $\underline{\mathcal{A}} \in \mathrm{Cond}(\mathrm{Ring})$
- $\mathcal{A}[-] : \mathrm{ED} \longrightarrow \mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}} = \{ \underline{\mathcal{A}}\text{-modules in } \mathrm{Cond}(\mathrm{Ab}) \}$

$$\mathcal{A}[S_1 \amalg S_2] = \mathcal{A}[S_1] \times \mathcal{A}[S_2]$$

- $\tau : S \to \mathcal{A}[S]$ nat. trans.

<u>Examples</u>

1) $\mathbb{Z}_{\blacksquare} = (\underline{\mathbb{Z}_{\blacksquare}} = \mathbb{Z},\ \mathbb{Z}_{\blacksquare}[S] = \mathbb{Z}[S]^{\blacksquare},\ S \to \mathbb{Z}[S]^{\blacksquare})$

2) $\mathbb{Z}_{p\blacksquare} = (\mathbb{Z}_p,\ \mathbb{Z}_{p\blacksquare}[S] = \mathbb{Z}_p[S]^{\blacksquare} = \lim \mathbb{Z}_p[S_i])$

2) $\mathbb{Z}_{p,\blacksquare} = \left(\mathbb{Z}_p ,\ \mathbb{Z}_{p,\blacksquare}[S] = \mathbb{Z}_p[S]^{\blacksquare} = \lim \mathbb{Z}_p[S_i] \right)$

3) $A \in \mathrm{Rng}$ $\quad (A,\mathbb{Z})_{\blacksquare} = \left(\underline{A} ,\ S \longmapsto \mathbb{Z}_{\blacksquare}[S] \otimes_{\mathbb{Z}} A \right)$

4) $A =$ fin. gen. $\mathbb{Z}$-alg

$$A_{\blacksquare} = \left(A ,\ A_{\blacksquare}[S] = \lim A[S_i] \right)$$

<u>Def</u> A pre-analytic ring $\mathcal{A}$ is <u>analytic</u> if $\mathcal{A}[-]$ satisfies $(*)$ in talk 5.

i.e.

$\forall\, C \in \mathrm{Cplx}_{\geq 0}\left(\mathrm{Mod}^{\mathrm{cond}}_{\underline{A}}\right)$ $\qquad C_i = \bigoplus_{k \in I_i} \mathcal{A}[J_k]$

$$\cdots \to C_i \to C_{i-1} \to \cdots \to C_1 \to C_0 \to 0$$

$$R\underline{\mathrm{Hom}}_{\underline{A}}(\mathcal{A}[S], C) \xrightarrow{\sim} R\underline{\mathrm{Hom}}_{\underline{A}}(\underline{A}[S], C)$$

<u>Prop 7.5</u> $\mathcal{A}$ analytic

Then we have Prop in Abstract framework.

<u>Def</u> $\mathrm{Mod}^{\mathrm{cond}}_{\underline{A}} \supset \mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}} = \{ M \in \mathrm{Mod}^{\mathrm{cond}}_{\underline{A}} \mid \forall S \in \mathrm{ED}$

$$\mathrm{Hom}_{\underline{A}}(\mathcal{A}[S], M) \simeq M(S) = \mathrm{Hom}_{\underline{A}}(\underline{A}[S], M) \}$$

- $\mathrm{Mod}^{\mathrm{cond}}_{\mathcal{A}}$ is abelian, stable under ...

- $\mathrm{Mod}_A^{\mathrm{cond}}$ is abelian, stable under ...

- $A[S]$ are compact projective generators of $\mathrm{Mod}_A^{\mathrm{cond}}$

- If $\underline{A}$ comm, sym. mon. $\otimes_{\underline{A}}$

- Similarly in $D(\mathrm{Mod}_A^{\mathrm{cond}})$

$$D(A) = D(\mathrm{Mod}_A^{\mathrm{cond}})$$

<u>Cor</u> $\mathbb{Z}[S]^{\blacksquare} = \lim \mathbb{Z}[S_i]$ is solid

$\Rightarrow$ $\mathbb{Z}_{\blacksquare} = (\mathbb{Z}, \mathbb{Z}[S]^{\blacksquare})$ is analytic

We have $\mathrm{Solid} = \mathrm{Mod}^{\mathrm{cond}}_{\mathbb{Z}_{\blacksquare}}$

$$\begin{array}{ccc} & & M^{\blacksquare} \\ & \nearrow & \| \\ M & \searrow & M \otimes_{\mathbb{Z}} \mathbb{Z}_{\blacksquare} \end{array}$$

§ Maps of analytic rings

<u>Def</u> A, B anal. rings

A <u>map of anal. rings</u> $f: A \to B$ is a map of cond. rings

$f : \underline{A} \to \underline{B}$ s.t. $\forall S \in ED$, $B[S]$ as $\underline{A}$-module lies in $\mathrm{Mod}_{\underline{A}}^{\mathrm{cond}}$

i.e. $\mathrm{Hom}_{\underline{A}}(A[S], B[S]) \simeq B[S](S)$

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & B[S] \\ & A[S] \arrow[u, dashed, "\exists!\ A\text{-linear}"'] \end{tikzcd}$$

<u>Prop</u> $f : A \to B$ map of analytic rings

1) $$\begin{tikzcd} \mathrm{Mod}_{\underline{A}}^{\mathrm{cond}} \arrow[r, "\otimes_{\underline{A}} \underline{B}"] \arrow[d, "\otimes_{\underline{A}} A"] & \mathrm{Mod}_{\underline{B}}^{\mathrm{cond}} \arrow[d, "\otimes_{\underline{B}} B"] \\ \mathrm{Mod}_{A}^{\mathrm{cond}} \arrow[r, dashed, "\exists!"', "\otimes_A B"] & \mathrm{Mod}_{B}^{\mathrm{cond}} \end{tikzcd}$$

2) Similarly on $D(-)$

Pf/ $\{A[S] \mid S \in ED\}$ form a family of compact proj. generators of $\mathrm{Mod}_A^{\mathrm{cond}}$

$$\begin{tikzcd} S \arrow[r] \arrow[d] & B[S] \\ A[S] \arrow[ur] & \end{tikzcd}$$

$\square$.

Prop $\mathbb{Z}_{p,\blacksquare}$ & $(A,\mathbb{Z})_\blacksquare$ are analytic rings

Pf/ Need: $\forall S \in E\mathcal{P}$, $\forall C$ good

$$R\underline{\mathrm{Hom}}_{\underline{A}}(A[S], C) \xrightarrow{\sim} R\underline{\mathrm{Hom}}_{\underline{A}}(\underline{A}[S], C)$$

$$\| $$

$$R\underline{\mathrm{Hom}}_{A}(A[S], C)$$

$$\|$$

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C)$$

$$\|$$

$\mathbb{Z}_\blacksquare$ analytic $\Rightarrow$ C solid

$$R\underline{\mathrm{Hom}}(\mathbb{Z}_\blacksquare[S], C)$$

$(A,\mathbb{Z})_\blacksquare[S] = \mathbb{Z}_\blacksquare[S] \otimes^L_{\mathbb{Z}} A$

$\otimes_{\mathbb{Z}} A \Rightarrow$ ok. $\square$

$(A, A^+) \rightsquigarrow (A, A^+)_\blacksquare$ analytic ring

$(\mathbb{Q}_p, \mathbb{Z}_p)$