Solid abelian groups

0. Motivation

Idea Want "completeness" condition on Cond (Ab)

In Cond (Ab) : Hom (A, B) , <u>RHom</u> (A, B) "good"

$-\otimes-$ on Cond (Ab) determined by

$$\mathbb{Z}[S] \otimes \mathbb{Z}[T] = \mathbb{Z}[S\times T]$$

eg $(\mathbb{Z}_p \otimes \mathbb{Z}_p)(*) = \mathbb{Z}_p \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ = too big

prefer: $\mathbb{Z}_p \hat{\otimes} \mathbb{Z}_p = \mathbb{Z}_p$ $\quad \mathbb{Z}_p \hat{\otimes} \mathbb{Z}_\ell = 0$ for $p \neq \ell$

Method Find Solid $\subset$ Cond (Ab) full subcat

closed under lim, colim, ext, s.t.

1) $\exists$ left adjoint $(-)^{■}$: Cond (Ab) $\longrightarrow$ Solid

$$M \longmapsto M^{■}$$

(<u>Solidification</u>)

2) $\exists$ complete tensor product $-\otimes^{■}-$ on Solid

(We will show : $\mathbb{Z}_p \otimes^{■} \mathbb{Z}_\ell = \begin{cases} \mathbb{Z}_p \\ 0 \end{cases}$)

In fact, Solid $\subset$ Cond (Ab) is the abelian subcat. gen. by

Compact proj $\mathbb{Z}[S]^{■}$, $S \in ED$

& have nice derived functor of $(-)^{■}$

$$L(-)^{■} : D(\mathrm{Cond}(Ab)) \longrightarrow D(\mathrm{Solid})$$

left adj. to $D(\mathrm{Solid}) \subset D(\mathrm{Cond}(Ab))$

How to produce Solid?

1) Abstract framework

2) Verify it for Cond(Ab)

1. Abstract framework

Prop $\mathcal{A}$ = ab. cat. gen. by compact proj under colimits

$\mathcal{A}^{cp} \subset \mathcal{A}$ full subcat of compact proj

$F : \mathcal{A}^{cp} \to \mathcal{A}$ additive + nat. tr. $\mathrm{id}_{\mathcal{A}^{cp}} \to F$

i.e. $X \to F(X)$ $\forall X \in \mathcal{A}^{cp}$

Suppose

(*) $\left[\begin{array}{l} \forall C \in \mathcal{A}^{cp},\ \forall M \in D(\mathcal{A}) \text{ of the form} \\ \quad \cdots \longrightarrow \bigoplus F(M_i) \longrightarrow \bigoplus F(N_j) \to 0 \\ R\mathrm{Hom}(F(C), M) \xrightarrow{\sim} R\mathrm{Hom}(C, M) \end{array} \right]$

Define $\mathcal{A}_F = \{ M \in \mathcal{A} \mid \mathrm{Hom}(F(C), M) = \mathrm{Hom}(C, M)$
$\forall C \in \mathcal{A}^{cp} \}$

$D_F(A) := \{ M \in D(A) \mid R\mathrm{Hom}(F(C), M) = R\mathrm{Hom}(C, M) \quad \forall C \in A^{cp} \}$

Then 1) $A_F \subset A$ ab. subcat closed under lim, colim, ext gen. by compact proj $\{ F(C) \mid C \in A^{cp} \}$

2) $A_F \subset A$ has a left adjoint $L : A \to A_F$ which is the unique colimit-preserving extension of $F : A^{cp} \to A_F \subset A$

3) $D(A_F) \to D(A)$ fully faithful with essential image $D_F(A)$, & has a left adjoint $D(A) \to D(A_F)$ which is left derived of L

4) $C \in D(A)$ belongs to $D_F(A) \Leftrightarrow \forall i,\ H^i(C) \in A_F$

$A = \mathrm{Cond}(Ab)$ $\qquad A^{cp} = \{ \mathbb{Z}[S] \mid S \in ED \}$

Refinement If 1) A has $\otimes$, commuting with colimit in each var

2) $(*)$ holds for $R\underline{\mathrm{Hom}}$ not just $R\mathrm{Hom}$

then 1) $\exists\ \overset{\blacksquare}{\otimes}$-structure on A_F s.t. $M \overset{\blacksquare}{\otimes} N = L(M \otimes N)$

$L : (A, \otimes) \to (A_F, \overset{\blacksquare}{\otimes})$ sym. mon.

i.e. $L(M \otimes N) = L(M) \overset{\blacksquare}{\otimes} L(N)$

2) Also holds in $D(-)$

2. Verify for Cond(Ab)

$\mathcal{A} = \mathrm{Cond(Ab)}$ $\qquad \mathcal{A}^{cp} = \{\mathbb{Z}[S] \mid S \in ED\}$

Need: Functor $F : \mathcal{A}^{cp} \to \mathcal{A}$ + nat. trans $\mathrm{id} \to F$

Def - $S = \lim S_i \in ED$ (finite)

$$\mathbb{Z}[S]^{\blacksquare} = \lim \mathbb{Z}[S_i] \in \mathrm{Cond(Ab)}$$

- $F(\mathbb{Z}[S]) = \mathbb{Z}[S]^{\blacksquare}$

- $S \to \mathbb{Z}[S]^{\blacksquare}$ induces $\mathbb{Z}[S] \to \mathbb{Z}[S]^{\blacksquare}$ $\Rightarrow$ $\mathrm{id} \to F$

  (injection)

Rk S finite $\quad \mathbb{Z}[S] \simeq \mathbb{Z}[S]^{\blacksquare} \simeq \bigoplus_S \mathbb{Z}$

$(*)$: $\quad R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, M) \xrightarrow{\sim} R\mathrm{Hom}(\mathbb{Z}[S], M)$

M = right bounded, termwise $\bigoplus \mathbb{Z}[S_i]^{\blacksquare}$

$$\mathrm{RHS} = R\mathrm{Hom}(\mathbb{Z}[S], M) = \mathrm{Hom}(S, M) = M(S)$$

$$\begin{aligned}\mathrm{LHS} = \mathbb{Z}[S]^{\blacksquare} = \lim \mathbb{Z}[S_i] &= \lim \underline{\mathrm{Hom}}(C(S_i,\mathbb{Z}),\mathbb{Z}) \\ &= \underline{\mathrm{Hom}}(\mathrm{colim}\, C(S_i,\mathbb{Z}), \mathbb{Z}) \\ &= \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z})\end{aligned}$$

$\Rightarrow$ the underlying abelian gp of $\mathbb{Z}[S]^{\blacksquare}$ is $\mathrm{Hom}(C(S,\mathbb{Z}),\mathbb{Z})$

$\Rightarrow$ the underlying abelian gp of $\mathbb{Z}[S]^{\blacksquare}$ is $\mathrm{Hom}(C(S,\mathbb{Z}),\mathbb{Z})$

= space of $\mathbb{Z}$-valued measures on S

<u>Th</u> (Speckov) $C(S,\mathbb{Z}) = \bigoplus_I \mathbb{Z}$ is a free abelian group

<u>Cor</u> $\mathbb{Z}[S]^{\blacksquare} = \prod_I \mathbb{Z}$

Baby case of $(*)$: $M = \bigoplus_i \mathbb{Z}[S_i]^{\blacksquare}$ in degree 0

$$= \bigoplus_i \prod \mathbb{Z}$$

Want: $R\mathrm{Hom}(\prod \mathbb{Z}, \bigoplus \prod \mathbb{Z}) \simeq R\mathrm{Hom}(\mathbb{Z}[S], \bigoplus \prod \mathbb{Z})$

RHS $= (\bigoplus \prod \mathbb{Z})(S) = \bigoplus \prod C(S,\mathbb{Z}) = \bigoplus \prod \bigoplus \mathbb{Z}$ in degree 0

LHS: $R\mathrm{Hom}(\prod \mathbb{Z}, \prod \mathbb{Z}) = \prod R\mathrm{Hom}(\prod \mathbb{Z}, \mathbb{Z}) = \prod \bigoplus \mathbb{Z}$

<u>Problem</u>: pull out $\bigoplus$

Trick: $0 \to \prod \mathbb{Z} \to \prod \mathbb{R} \to \underbrace{\prod \mathbb{R}/\mathbb{Z}}_{} \to 0$

pseudo-coherent $\Rightarrow$ commute with $\bigoplus$

A cpt ab gp $\overset{\text{Deligne}}{\rightsquigarrow}$ $\mathbb{Z}[A] \rightsquigarrow \mathbb{Z}[S]$ $S \in ED$

Remains to control $\prod \mathbb{R}$

<u>Lemma</u> $R\mathrm{Hom}(\prod \mathbb{R}, \bigoplus \prod \mathbb{Z}) = 0$

D/ $R\mathrm{Hom}(\mathbb{R},\mathbb{Z}) = 0 \Rightarrow R\mathrm{Hom}(\prod \mathbb{R}, \bigoplus \prod \mathbb{Z})$

$= R\mathrm{Hom}(\prod \mathbb{R}, R\mathrm{Hom}(\mathbb{R}, \bigoplus \prod \mathbb{Z}))$

b/ $R\mathrm{Hom}(\mathbb{R},\mathbb{Z})=0 \Rightarrow R\mathrm{Hom}(\prod\mathbb{R}, \bigoplus\prod\mathbb{Z})$

$$= R\mathrm{Hom}_{\mathbb{R}}\Big(\prod\mathbb{R}, \underbrace{R\mathrm{Hom}(\mathbb{R}, \bigoplus\prod\mathbb{Z})}_{0}\Big) = 0. \qquad \square$$

3. Solid Abelian Group

Def 1) $\mathrm{Solid} \subset \mathrm{Cond}(\mathrm{Ab})$

$$\mathrm{Solid} = \{A \mid \forall S \in ED,\ \mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, A) \simeq \mathrm{Hom}(\mathbb{Z}[S], A)\}$$

2) $D\mathrm{Solid} \subset D(\mathrm{Cond}(\mathrm{Ab}))$

$$D\mathrm{Solid} = \{C \mid \_\_\_\_\_\_ \quad R\mathrm{Hom}(\quad, C) \simeq R\mathrm{Hom}(\quad, C)\}$$

Cor $\mathbb{Z}[S]^{\blacksquare}$ is solid in $\mathrm{Cond}(\mathrm{Ab})$ & $D(\mathrm{Cond}(\mathrm{Ab}))$