Locally compact abelian groups (LCA)

<u>Goal</u> - $A, B \in LCA$ compute $\mathrm{Ext}^i(\underline{A}, \underline{B})$

- $D(LCA) \hookrightarrow D(\mathrm{Cond}(Ab))$ fully faithful

<u>Recall</u> $X \in \mathrm{Top}(Ab)$ is <u>locally compact</u> if $\forall x \in X$ $\exists$ basis of compact Hausdorff nbh (not nec. open)

Eg: $\mathbb{R}^n$, $\mathbb{R}/\mathbb{Z} = \mathbb{T}$, $\mathbb{Z}_p$

$\mathbb{Q}$ is not.

<u>Structure thm</u> $\forall A \in LCA$

$A = A' \times \mathbb{R}^n$ $\qquad 0 \to \underset{\text{cpt}}{A^2} \to A^1 \to \underset{\text{discrete}}{A^3} \to 0$

<u>Pontryagin duality</u>

$$\mathbb{D} : LCA^{op} \longrightarrow LCA \qquad \mathbb{D}^2 = id$$
$$A \longmapsto \mathrm{Hom}(A, \mathbb{T})$$

<u>Ex</u> $\mathbb{D}(\mathbb{T}) = \mathbb{Z}$ $\qquad \mathbb{D}(\mathbb{R}) = \mathbb{R}$

$\mathbb{D}(\mathbb{F}_p) = \mathbb{F}_p$ $\qquad$ Fourier transform

$\mathbb{D}$

$$\text{discrete} \underset{\mathbb{D}}{\rightleftarrows} \text{compact}$$

$\mathrm{Hom}(A,B)$ with compact open topology

Back to condensed

$\underline{\mathrm{Hom}}$ = sheaf Hom in $\mathrm{Cond}(\mathrm{Ab})$

Prop $A, B \in LCA$ Hausdorff, A compactly generated. $\quad \underline{\mathrm{Hom}}(\underline{A}, \underline{B}) = \underline{\mathrm{Hom}(A,B)}$

Lemma 1 X, Y Hausdorff, X cpt. generated

$$\underline{\mathrm{Hom}}(\underline{X}, \underline{Y}) = \underline{\mathrm{Hom}(X,Y)} \in \mathrm{Cond}(\mathrm{Set})$$

Lemma 2 Given $M, N \in \mathrm{Cond}(\mathrm{Ab})$

as condensed sets

$$\underset{\substack{\cap \\ \mathrm{Cond}(\mathrm{Ab})}}{\underline{\mathrm{Hom}}(M,N)} = \ker\{\underline{\mathrm{Hom}}(M,N) \to \underline{\mathrm{Hom}}(M \times M, N)\}$$

$$f \longmapsto \big((m_1, m_2) \mapsto f(m_1+m_2) - f(m_1) - f(m_2)\big)$$

P/ Lemma 1: $\underline{\mathrm{Hom}}(\underline{X}, \underline{Y})(S) = \mathrm{Hom}(S, \underline{\mathrm{Hom}}(\underline{X}, \underline{Y}))$

$$= \mathrm{Hom}(\underline{X} \times \underline{S}, \underline{Y}) = \mathrm{Hom}_{cts}(X \times S, Y)$$

$$= \mathrm{Hom}_{cts}(S, \mathrm{Hom}(X,Y)) \simeq \mathrm{Hom}(\underline{S}, \underline{\mathrm{Hom}}(X,Y))$$

$$= \underline{\mathrm{Hom}}(X,Y)(S).$$

Lemma 2: Need:

$$\mathrm{Hom}_{\mathrm{Cond}(\mathrm{Ab})}(\mathbb{Z}[S], \underline{\mathrm{Hom}}(M,N))$$

$$= \mathrm{Hom}_{\mathrm{Cond}(\mathrm{Ab})}(M \otimes \mathbb{Z}[S], N)$$

$$= \mathrm{Ker}\{\mathrm{Hom}(M \times S, N) \to \mathrm{Hom}(M \times M \times S, N)\}$$

Resolution:

$$\begin{tikzcd}
\mathbb{Z}[M \times M] \arrow[r] \arrow[d, "\otimes \mathbb{Z}[S]"'] & \mathbb{Z}[M] \arrow[r] & M \arrow[r] & 0 \\
\mathbb{Z}[M \times M \times S] \arrow[r] \arrow[d, "\mathrm{Hom}(-,N)"'] & \mathbb{Z}[M \times S] \arrow[r] & M \otimes \mathbb{Z}[S] \arrow[r] & 0 \\
\text{QED} & & &
\end{tikzcd}$$

$$[(x,y)] \mapsto [x+y] - [x] - [y]$$

$\square$.

<u>Th</u> (key computation) $\quad A = \prod_I \mathbb{T}$

1) M discrete $\qquad R\underline{\mathrm{Hom}}(\underline{A}, \underline{M}) = \bigoplus_I \underline{M}[-1]$

where $M[-1] = R\underline{\mathrm{Hom}}(\mathbb{Z}[1], \underline{M}) \qquad \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to \mathbb{Z}[1]$

$$\begin{tikzcd}
R\underline{\mathrm{Hom}}(\mathbb{Z}[1], \underline{M}) \arrow[d] & \\
R\underline{\mathrm{Hom}}(\underline{\mathbb{T}}, \underline{M}) \arrow[r, "p_i^*"] & R\underline{\mathrm{Hom}}(\underline{A}, M)
\end{tikzcd}$$

2) $R\underline{\mathrm{Hom}}(\underline{A}, \underline{\mathbb{R}}) = 0$

$\underline{\text{Cor}}$ $R\mathrm{Hom}(\underline{\mathbb{R}}, \underline{\mathbb{R}}) = \underline{\mathbb{R}}$

Grothendieck ss

$\underline{\text{Rk}}$ $H^i(R^j \underline{\mathrm{Hom}}(\underline{A}, \mathbb{R})) \Rightarrow \mathrm{Ext}^i(\mathbb{Z}[A], \mathbb{R})$

But $H^0(S, \mathbb{R}) \neq 0$

(Deligne resolution)

$\underline{\text{Main tool}}$: $\exists$ functorial resolution , $\forall M \in G_p$

$$\cdots \to \mathbb{Z}[M^3] \oplus \mathbb{Z}[M^2] \to \mathbb{Z}[M \times M] \to \mathbb{Z}[M] \to M \to 0$$

$$\bigoplus_j \mathbb{Z}[M^{r_{ij}}]$$

$$[(x,y)] \mapsto [x+y] - [x] - [y]$$

r_{ij} finite, indep of M

$\underline{\text{Cor}}$ $A, M \in \mathrm{Cond}(Ab)$ $\quad S \in ED$

$M \mapsto A$

$\otimes \mathbb{Z}[S]$

$R\mathrm{Hom}(-, M)$

$$E_1^{i_1, i_2} = \prod_{j=1}^{n_{i_1}} H^{i_2}(A^{r_{i_1,j}} \times S, M) \Rightarrow \underline{\mathrm{Ext}}^{i_1+i_2}(A, M)(S)$$

$\underline{\text{Pf of Thm}}$ $F_\cdot(A) \to A$ Deligne resolution

$R\mathrm{Hom}(\underline{A}, \underline{\mathbb{R}})(S)$ computed by

$$0 \to \bigoplus C(A^{r_{0j}} \times S, \mathbb{R}) \to \bigoplus C(A^{r_{ij}} \times S, \mathbb{R}) \to \cdots$$

complex of Banach spaces

Need to show it is exact (omitted) $\square$.