Cohomology

$S \in CH$ 1) Singular coh $H^*_{sing}(S, \mathbb{Z})$

2) Cech cohomology $S = \bigcup S_\alpha$

$$\Gamma(S_{\alpha_{i_1}} \cap \cdots S_{\alpha_{i_n}}, \mathbb{Z}) \Rightarrow \check{H}^\bullet(S, \mathbb{Z})$$

$$S_0 = \coprod_\alpha S_\alpha \qquad S_0 \to S$$

$$S_1 = S_0 \times_S S_0 \qquad S_1 \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} S_0 \qquad (\to S_n \to S_{n-1} \to \cdots \to S_1 \to S_0) \to S$$

$$S_n = \underbrace{S_0 \times_S \cdots \times_S S_0}_{n+1 \text{ times}} \qquad S_n \overset{n+1}{\underset{\vdots}{\rightrightarrows}} S_{n-1}$$

$$\check{H}^i(S, \mathbb{Z}) = H^i\Big(\operatorname*{colim}_{S_\bullet \to S} R\Gamma(S_\bullet, \mathbb{Z})\Big)$$

3) Sheaf cohomology $H^*_{sheaf}(S, \underline{\mathbb{Z}})$

Comparison: (2)=(3) for paracompact Hausdorff

(1) bad for profinite S

$$H^0_{sing}(S, \mathbb{Z}) = \mathrm{Hom}_{Set}(S, \mathbb{Z})$$

$$\check{H}^0(S, \mathbb{Z}) = \mathrm{Hom}_{top}(S, \mathbb{Z})$$

$$\underline{\mathbb{Z}} : S \mapsto C^0(S, \mathbb{Z})$$

Def $H^i_{cond}(S, \mathbb{Z}) = \mathrm{Ext}^i(\underbrace{\mathbb{Z}[\underline{S}]}_{\in}, \underbrace{\underline{\mathbb{Z}}}_{\in})$ Cond(Ab)

($H^i_{cond}(S, -) = R^i\Gamma(S, -) : D(\mathrm{Cond}(Ab)) \to D(Ab)$)

Th ($\mathbb{Z}$) $\exists$ nat. func. iso

$$H^i_{sheaf}(S,\mathbb{Z}) = H^i_{cond}(S,\mathbb{Z})$$

Th ($\mathbb{R}$) $H^0_{cond}(S,\mathbb{R}) = C(S,\mathbb{R})$

$H^i_{cond}(S,\mathbb{R}) = 0 \quad i>0.$

Rk $H^i_{sheaf}(S,\mathbb{Z})\otimes\mathbb{R} = H^i_{sheaf}(S,\mathbb{R})$

fails for H^i_{cond}

Pf of ($\mathbb{Z}$): finite $\to$ profinite $\to$ general

- Finite: $H^i_{cond}(S,\mathbb{Z}) = H^i_{sheaf}(S,\mathbb{Z}) = \begin{cases} C(S,\mathbb{Z}) & i=0 \\ 0 & i\geq 1 \end{cases}$

- Profinite $S = \lim_j S_j$

• $\operatorname{colim}_j H^i_{sheaf}(S_j,\mathbb{Z}) \simeq H^i_{sheaf}(\underbrace{\lim_{\leftarrow j} S_j}_{=S}, \mathbb{Z})$

$\Rightarrow H^i_{sheaf}(S,\mathbb{Z}) = \begin{cases} \operatorname{colim}_j C(S_j,\mathbb{Z}) & i=0 \\ 0 & i>0 \end{cases}$

• $H^0_{cond}(S,\mathbb{Z}) = \operatorname{Hom}_{Cond(Ab)}(\mathbb{Z}[S],\mathbb{Z}) = \operatorname{Hom}_{top}(S,\mathbb{Z})$

For H^i, $i>0$, choose $S_\bullet \to S$ simplicial res. s.t. S_n ED.

$S = \lim S^j$ $\quad S^j$ finite $\quad$ finite $S_n^j \longrightarrow S^j$ finite

$S_n = \lim S_n^j$ $\quad S_n^j$ finite

$$\begin{array}{ccc} S_n^j & \longrightarrow & S^j \\ \downarrow & & \downarrow \\ S_n & \longrightarrow & S \end{array}$$

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For each j, $0 \to \Gamma(S^j, \mathbb{Z}) \to \Gamma(S^j_0, \mathbb{Z}) \to \Gamma(S^j_1, \mathbb{Z}) \to \cdots$ exact

$\operatorname{colim}_j(-) \Rightarrow 0 \to \operatorname{colim}_j \Gamma(S^j, \mathbb{Z}) \to \operatorname{colim}_j \Gamma(S^j_0, \mathbb{Z}) \to \cdots$ exact

$$\underbrace{\operatorname{colim}_j \Gamma(S^j, \mathbb{Z})}_{= \Gamma(S,\mathbb{Z})} \qquad \underbrace{\operatorname{colim}_j \Gamma(S^j_0, \mathbb{Z})}_{= \Gamma(S_0,\mathbb{Z})}$$

– General case

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$\mathrm{Cond}(\mathrm{Ab}) \simeq \mathrm{Sh}(\mathrm{CH})$

$S^{CH} = \{ V \subset S, V \in CH \}$

$\alpha : \mathrm{Sh}(S^{CH}) \to \mathrm{Sh}(S)$

$\mathcal{F} \longmapsto \alpha_* \mathcal{F}$

$U \subset S$ open $\quad \alpha_* \mathcal{F}(U) = \lim_{V \subset U} \mathcal{F}(V)$

α_* left exact

Need: $R\alpha_* \mathbb{Z} = \mathbb{Z}$

Stalks: $s \in S$ $\qquad (R\alpha_* \mathbb{Z})_s = \operatorname{colim}_{s \in U \text{ open}} R\Gamma(U, R\alpha_* \mathbb{Z})$

$= \operatorname{colim}_{s \in U \text{ open}} R\Gamma_{\mathrm{cond}}(U, \mathbb{Z})$

$= \operatorname{colim}_{s \in V \text{ closed nbh}} R\Gamma_{\mathrm{cond}}(V, \mathbb{Z})$

$S_0 \to S$ simplicial hypercover $\qquad S_n$ ED

$\Rightarrow \{S_n \times_S V\} \longrightarrow V$ simplicial hypercover

$\uparrow$ ED

$R\Gamma_{cond}(V, \mathbb{Z})$ is computed by $0 \to \Gamma(S_0 \times_S V, \mathbb{Z}) \longrightarrow \cdots$ exact

Take $\operatorname{colim}_{s \in V} \rightsquigarrow 0 \to \Gamma(S_0 \times_S \{s\}, \mathbb{Z}) \to \Gamma(S_1 \times_S \{s\}, \mathbb{Z}) \to \cdots$ exact

But $\{S_n \times_S \{s\}\}$ = simplicial hypercover of S

$\Rightarrow R\Gamma_{cond}(\{s\}, \mathbb{Z}) = \mathbb{Z}_s$

$\Rightarrow R\alpha_* \mathbb{Z} = \mathbb{Z}$ $\square$