

Def (Johnstone 02 § C2.1)  $\mathcal{C} \in \text{Cat}$

A coverage on  $\mathcal{C}$  is  $\forall U \in \mathcal{C}$ , a collection of families  $\{f_i: U_i \rightarrow U\}$

Called covering families

If  $\{U_i \xrightarrow{f_i} U\}$  CF,  $g: V \rightarrow U$

$$\exists \text{ CF } \{V_j \xrightarrow{h_j} V\} \quad \begin{array}{ccc} V_j & \xrightarrow{k} & U_i \\ h_j \downarrow & & \downarrow f_i \\ V & \xrightarrow{g} & U \end{array}$$

A site is a category + coverage

Def A functor  $\mathcal{C}^{\text{op}} \rightarrow \text{Sets}$  is a sheaf if

$$\forall \text{ CF } \{f_i: U_i \rightarrow U\}$$

$$\forall (S_i \in A(U_i)) \quad \underline{\text{compatible}}, \text{ i.e. } \forall \begin{array}{ccc} V & \xrightarrow{g} & U_i \\ h \downarrow & & \downarrow f_i \\ U_j & \xrightarrow{f_j} & U \end{array}$$

$$g^* S_i = h^* S_j$$

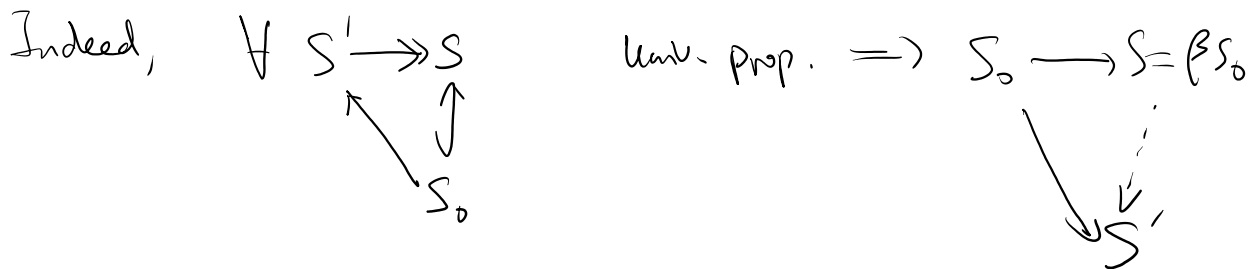
$$\exists! S \in A(U) \quad f_i^*(S) = S_i$$

Ex 2.5  $S_0 = \text{discrete set}$

$S = \beta S_0$  Stone-Čech compactification

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Then  $S \in ED$ .

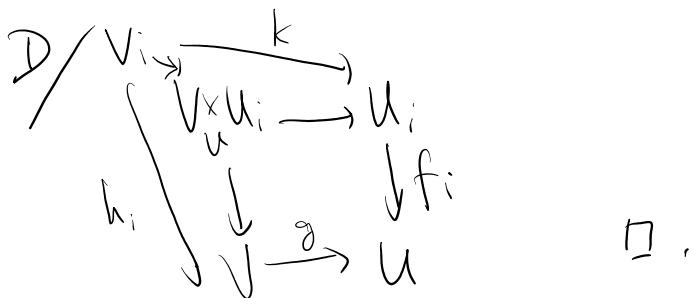


In part,  $\forall S \in CH$   $S' = \beta S_{\text{discrete}} \twoheadrightarrow S$   
 $|S'| < 2^{2^{|S|}}$

$\kappa =$  uncountable strong limit cardinal

$$\Rightarrow \forall \lambda < \kappa, 2^\lambda < \kappa \Rightarrow 2^{2^\lambda} < \kappa \Rightarrow \text{ok.}$$

Prop  $ED_\kappa$  + coverage given by finite family of jointly surjective maps is a site.



Prop A functor  $T: ED_2^{op} \rightarrow \text{Sets}$  is a sheaf on  $ED_2$

$$\Leftrightarrow T(\coprod S_i) \cong \prod T(S_i)$$

sends finite disj. union to finite product

D/  $f: S' \rightarrow S$   $e: S \rightarrow S'$   $SET(S')$  compatible

$$\forall S'' \xrightarrow{g} S' \\ \begin{array}{ccc} h \downarrow & & \downarrow f \\ S' & \xrightarrow{f} & S \end{array}$$

$$g^*(s) = h^*(s)$$

$$\begin{array}{ccc} S' \times_S S' & \rightarrow & \tilde{S} \\ & \searrow & \downarrow \\ & & S' \times_S S' \end{array}$$

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{g} & S' \\ \downarrow h & & \downarrow f \\ S' & \xrightarrow{f} & S \end{array}$$

$$\begin{array}{ccccc} \tilde{S} & \xrightarrow{g} & S' & & \\ \downarrow h & & \downarrow f & & \\ S' & \xrightarrow{f} & S & & \\ \uparrow e \circ f & & & & \\ S' & & & & \end{array}$$

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{g} & S \\ \downarrow p' & & \downarrow p \\ S' \times_S S' & & S \end{array}$$

$$g^*(s) = h^*(s)$$

$$(g \circ g)^*(s) = (h \circ g)^*(s)$$

$$g \circ g = p_2 \circ p_1 \circ g = p_2 \circ p' = g$$

$$h \circ g = p_1 \circ p_1 \circ g = p_1 \circ p' = e \circ f \circ h$$

i.e.  $g^*(s) = h^*(e \circ f \circ h)^*(s) = h^*(s)$

$$f^*(e^*(s)) = s$$

$$s_0 = e^*(s)$$

□

Prop 2.5  $\text{Cond}_k(\text{Sets}) \xrightarrow[\leftarrow]{\text{restriction}} \text{Sh}(ED_k)$  is an equivalence.

D/  $T \in \text{Sh}(ED_k) \quad \forall S \in \text{Prof} \quad \tilde{S} \rightarrow S$

$T(S)$  is the equalizer  $\tilde{S} \rightarrow \tilde{S} \times_S \tilde{S} \rightarrow \tilde{S}$

$T(S)$  is the equalizer  $S \rightarrow \tilde{S} \times_S \tilde{S} \rightarrow \tilde{S}$   
of  $T(\tilde{S}) \rightrightarrows T(\tilde{S})$   $\searrow \downarrow \downarrow$   
 $\tilde{S} \rightarrow S$   $\square$ .

Th 2-2  $\text{Cond}_k(\text{Ab})$  is an abelian category with

- |                             |   |
|-----------------------------|---|
| (AB3) <sup>S</sup> limits   | (AB5) filtered colimits are exact                   |
| (AB3) colimits              | (AB6) $\forall J, (I_j)_{j \in J}$<br>filtered cat. |
| (AB4) <sup>*</sup> products | $I_j \rightarrow \text{Cond}_k(\text{Ab})$          |
| (AB4) direct sums           | $i \mapsto M_i$                                     |

$$\text{colim}_i \prod_j M_{ij} \xrightarrow{\cong} \prod_j \text{colim}_i M_{ij}$$

Moreover,  $\text{Cond}_k(\text{Ab})$  is generated by compact projective objects

i.e.  $\text{Hom}(S, -)$  commutes with all limits & is exact

D/  $\text{Ab} \rightarrow \text{Sets}$  has a left adjoint  $T \rightarrow \mathbb{Z}[T]$

$\Rightarrow \text{Cond}_k(\text{Ab}) \rightarrow \text{Cond}_k(\text{Sets})$  has a left adjoint  $T \rightarrow \mathbb{Z}[T]$

$$\forall S \in \text{ED}_k \quad \underline{\Sigma}(S') \cong \text{Hom}(S', S) \quad \mathbb{Z}[T](S) = \mathbb{Z}[T(S)]$$

$$\forall M \in \text{Cond}_k(\text{Ab}) \quad \text{Hom}(\mathbb{Z}[\underline{\Sigma}], M) \cong \text{Hom}(\underline{\Sigma}, M) = M(S)$$

$$- M=0 \Leftrightarrow \forall S \in \text{ED}_k, M(S)=0$$

- any  $M$  has a surjection from a direct sum of  $\mathbb{Z}[\underline{\Sigma}]$

Indeed, by Zorn Lemma,  $\exists$  maximal subobject  $M' \subset M$

$$+ \quad \bigoplus_S \mathbb{Z}[\Sigma] \rightarrow M'$$

If  $M/M' \neq 0$ ,  $\exists S \in \mathcal{E}D_{\mathbb{Z}} \quad M/M'(S) \neq 0$

$\text{Hom}(\mathbb{Z}[\Sigma], M/M') \neq 0$ , contradiction.

$$\text{Cond}(Ab) := \text{cdim}_{\mathbb{Z}} \text{Cond}_{\mathbb{Z}}(Ab)$$

-  $M \otimes N = \text{sheafification of } S \mapsto M(S) \otimes N(S)$

$$\mathbb{Z}[T] \otimes \mathbb{Z}[T] = \mathbb{Z}[T \times T] \quad T \in \text{Cond}(\text{Sets})$$

$\mathbb{Z}[T]$  is flat

$$D(\text{Cond}(Ab)) \quad , \quad \otimes^L$$

$$\text{Hom}(P, \underline{\text{Hom}}(M, N)) \simeq \text{Hom}(P \otimes M, N)$$

$$\rightsquigarrow \text{R}\underline{\text{Hom}}(-, -)$$

$$\text{Hom}(P, \text{R}\underline{\text{Hom}}(M, N)) \simeq \text{Hom}(P \otimes^L M, N)$$