<u>Def</u> ([Johnstone 02 § C2.1] $\mathcal{C} \in \mathrm{Cat}$

A <u>coverage</u> on $\mathcal{C}$ is $\forall\, U \in \mathcal{C}$, a collection of families

$$\{f_i : U_i \to U\}$$

called covering families

If $\{U_i \xrightarrow{f_i} U\}$ CF, $g : V \to U$

$\exists$ CF $\{V_j \xrightarrow{h_j} V\}$

$$\begin{tikzcd} V_j \arrow[r, "k"] \arrow[d, "h_j"'] & U_i \arrow[d, "f_i"] \\ V \arrow[r, "g"] & U \end{tikzcd}$$

A <u>site</u> is a category + coverage

<u>Def</u> A functor $\mathcal{C}^{op} \xrightarrow{A} \mathrm{Sets}$ is a <u>sheaf</u> if

$\forall$ CF $\{f_i : U_i \to U\}$

$\forall\ (s_i \in A(U_i))$ <u>compatible</u>, i.e. $\forall$

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "h"'] & U_i \arrow[d, "f_i"] \\ U_j \arrow[r, "f_j"] & U \end{tikzcd}$$

$$g^* s_i = h^* s_j$$

$\exists!\, s \in A(U)$ $f_i^*(s) = s_i$

<u>Ex 2.5</u> S_0 = discrete set

$S = \beta S_0$ Stone-Čech compactification

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Then $S \in ED$.

Indeed, $\forall$
$$\begin{tikzcd} S' \arrow[r, two heads] & S \\ & S_0 \arrow[ul] \arrow[u, hook] \end{tikzcd}$$
univ. prop. $\Rightarrow$
$$\begin{tikzcd} S_0 \arrow[r] \arrow[dr] & S = \beta S_0 \arrow[d, dashed] \\ & S' \end{tikzcd}$$

In part, $\forall S \in CH$ $\quad S' = \beta S_{\text{discrete}} \twoheadrightarrow S$

$|S'| < 2^{2^{|S|}}$

κ = uncountable strong limit cardinal

$\Rightarrow \forall \lambda < \kappa,\ 2^\lambda < \kappa \Rightarrow 2^{2^\lambda} < \kappa \quad \Rightarrow$ ok.

<u>Prop</u> ED_κ + coverage given by finite family of jointly surjective maps is a site.

Pf/
$$\begin{tikzcd} V_i \arrow[r] \arrow[rr, bend left, "k"] \arrow[dr, bend right, "h_i"'] & V \times_U U_i \arrow[r] \arrow[d] & U_i \arrow[d, "f_i"] \\ & V \arrow[r, "g"] & U \end{tikzcd}$$
$\square$.

<u>Prop</u> A functor $T: ED_\kappa^{op} \to Sets$ is a sheaf on ED_κ

$\Leftrightarrow T(\coprod S_i) \simeq \prod T(S_i)$

sends finite disj. union to finite product

D/ $f : S' \twoheadrightarrow S$ $\quad e : S \to S'$

$s \in T(S')$ compatible

$$\begin{tikzcd} \forall\ S'' \arrow[r, "g"] \arrow[d, "h"'] & S' \arrow[d, "f"] \\ S' \arrow[r, "f"'] & S \end{tikzcd} \qquad g^*(s) = h^*(s)$$

$$\begin{tikzcd} S' \times_S S' \arrow[r] \arrow[dr] & \tilde{S} \arrow[d] \\ & S' \times_S S' \end{tikzcd}$$

$$\begin{tikzcd} \tilde{S} \arrow[drr, bend left, "g"] \arrow[dr, "p"] \arrow[ddr, "h"'] & & \\ & S' \times_S S' \arrow[r, "p_2"] \arrow[d, "p_1"] & S' \arrow[d, "f"] \\ & S' \arrow[r, "f"'] & S \end{tikzcd}$$

$$\begin{tikzcd} \tilde{S} \arrow[r, dashed, "p'"] \arrow[rr, bend left, "g"] \arrow[d, "h"'] & S' \times_S S' \arrow[r, "p_2"] \arrow[d, "p_1"] & S' \arrow[d, "f"] \\ S' \arrow[r, "e \circ f"] \arrow[rr, bend right, "f"'] & S' \arrow[r, "f"] & S \end{tikzcd}$$

$$\begin{tikzcd} \tilde{S} \arrow[r, "q"] \arrow[dr, "p'"'] & \tilde{S} \arrow[d, shift left, "p"] \\ & S' \times_S S' \arrow[u, shift left] \end{tikzcd}$$

$g^*(s) = h^*(s)$ $\qquad (g \circ q)^*(s) = (h \circ q)^*(s)$

$$g q = p_2 p q = p_2 p' = g$$

$$h q = p_1 p q = p_1 p' = e f h$$

i.e. $g^*(s) = h^* f^* e^*(s) = h^*(s)$

$f^* e^*(s) = s$ $\qquad s_0 = e^*(s)$. $\qquad \square$.

<u>Prop 2.5</u> $\mathrm{Cond}_\kappa(\mathrm{Sets}) \underset{\longleftarrow}{\xrightarrow{\text{restriction}}} \mathrm{Sh}(\mathrm{ED}_\kappa)$ is an equivalence.

D/ $T \in \mathrm{Sh}(\mathrm{ED}_\kappa)$ $\quad \forall S \in \mathrm{Prof}$ $\quad \tilde{S} \twoheadrightarrow S$

$T(S)$ is the equalizer $\tilde{\tilde{S}} \twoheadrightarrow \tilde{S} \times_S \tilde{S} \longrightarrow \tilde{S}$

$$\begin{tikzcd} T(S) \text{ is the equalizer of } T(\tilde{S}) \rightrightarrows T(\tilde{\tilde{S}}) & S \arrow[r] \arrow[dr] & \tilde{S}\times_S \tilde{S} \arrow[r] \arrow[d] & \tilde{S} \arrow[d] \\ & & \tilde{S} \arrow[r] & S \end{tikzcd}$$

$\square$.

Th 2.2 $\mathrm{Cond}_\kappa(Ab)$ is an abelian category with

(AB3)* limits

(AB3) colimits

(AB4)* products

(AB4) direct sums

(AB5) filtered colimits are exact

(AB6) $\forall J$, $(I_j)_{j\in J}$ filtered cat.

$I_j \to \mathrm{Cond}_\kappa(Ab)$

$i \mapsto M_i$

$$\operatorname{colim}_i \prod_j M_{ij} \xrightarrow{\sim} \prod_j \operatorname{colim}_i M_{ij}$$

Moreover, $\mathrm{Cond}_\kappa(Ab)$ is generated by compact projective objects

(i.e. $\mathrm{Hom}(S,-)$ commutes with all colimits & is exact)

D/ $Ab \to Sets$ has a left adjoint $T \mapsto \mathbb{Z}[T]$

$\Rightarrow$ $\mathrm{Cond}_\kappa(Ab) \to \mathrm{Cond}_\kappa(Sets)$ has a left adjoint $T \mapsto \mathbb{Z}[T]$

$\mathbb{Z}[T](S) = \mathbb{Z}[T(S)]$

$\forall S \in ED_\kappa$ $\quad \underline{S}(S') \simeq \mathrm{Hom}(S', S)$

$\forall M \in \mathrm{Cond}_\kappa(Ab)$ $\quad \mathrm{Hom}(\mathbb{Z}[\underline{S}], M) \simeq \mathrm{Hom}(\underline{S}, M) = M(S)$

- $M = 0 \Leftrightarrow \forall S \in ED_\kappa,\ M(S) = 0$

- any M has a surjection from a direct sum of $\mathbb{Z}[\underline{S}]$

Indeed, by Zorn Lemma, $\exists$ maximal subobject $M' \subset M$

$$+ \quad \bigoplus_{S} \mathbb{Z}[\underline{S}] \twoheadrightarrow M'$$

If $M/M' \neq 0$, $\exists\, S \in ED_\kappa$ $\quad M/M'(S) \neq 0$

$\mathrm{Hom}(\mathbb{Z}[\underline{S}], M/M') \neq 0$, contradiction.

$\mathrm{Cond}(\mathrm{Ab}) := \operatorname{colim}_{\kappa} \mathrm{Cond}_\kappa(\mathrm{Ab})$

- $M \otimes N$ = sheafification of $S \longmapsto M(S) \otimes N(S)$

$\mathbb{Z}[T] \otimes \mathbb{Z}[T] = \mathbb{Z}[T \times T] \qquad T \in \mathrm{Cond}(\mathrm{Sets})$

$\mathbb{Z}[T]$ is flat

$D(\mathrm{Cond}(\mathrm{Ab}))$, $\otimes^L$

$$\mathrm{Hom}(P, \underline{\mathrm{Hom}}(M, N)) \simeq \mathrm{Hom}(P \otimes M, N)$$

$$\rightsquigarrow R\underline{\mathrm{Hom}}(-, -)$$

$$\mathrm{Hom}(P, R\underline{\mathrm{Hom}}(M, N)) \simeq \mathrm{Hom}(P \otimes^L M, N)$$