1. Condensed sets

1. Sites and sheaves

Def $\mathcal{C}$ = cat $X \in \mathcal{C}$

- A *sieve* S on X is a subfunctor of $\mathrm{Hom}(-, X)$

- S sieve on X, $f: Y \to X$ *$f^* S$* sieve on Y

$$f^* S(Z) := \{ Z \xrightarrow{g} Y \mid f \circ g \in S(Z) \}$$

- $\mathfrak{X} = \{ X_i \xrightarrow{f_i} X \}_{i \in I}$

$$S_{\mathfrak{X}}(Y) = \{ Y \xrightarrow{g} X \mid g \text{ factors through some } f_i \}$$

- $F: \mathcal{C}^{op} \to \text{Sets}$, S sieve on X,

F satisfies the sheaf condition w.r.t. S if F satisfies the following equivalent conditions:

1) $\mathrm{Nat}(\mathrm{Hom}(-, X), F) \to \mathrm{Nat}(S, F)$ bijection

2) $$F(X) \xrightarrow{e} \prod F(\mathrm{dom}\, f) \underset{b}{\overset{a}{\rightrightarrows}} \prod F(\mathrm{dom}\, g)$$

2) $$F(X) \xrightarrow{e} \prod_{f \in S} F(\mathrm{dom}\, f) \underset{p}{\overset{a}{\rightrightarrows}} \prod_{\substack{f,g \\ f \in S \\ \mathrm{cod}\, g = \mathrm{dom}\, f}} F(\mathrm{dom}\, g)$$

is an equalizer diagram

$f : \mathrm{dom}\, f \to X \qquad e = f^* : F(X) \to F(\mathrm{dom}\, f)$

· $x = (x_f)_{f \in S} \qquad a(x_f)_{f,g} = x_{f \circ g} \qquad g : \mathrm{dom}\, g \to \mathrm{dom}\, f$

$p(x_f)_{f,g} = g^*(x_f) \qquad g^* : F(\mathrm{dom}\, f) \to F(\mathrm{dom}\, g)$

$$\boxed{\begin{array}{l} S \longrightarrow F \\ f \longmapsto x_f \end{array}}$$

If further S gen. by a family $\mathfrak{X} = \{X_i \xrightarrow{f_i} X\}_{i \in I}$

and $X_i \times_X X_j$ exists

Then 1) & 2) are equivalent to

3) The diagram

$$F(X) \xrightarrow{e} \prod_{i \in I} F(X_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(X_i \times_X X_j)$$ is equalizer

<u>Def</u> $\mathcal{C} \in \mathrm{Cat}$ A <u>grothendieck topology</u> J on $\mathcal{C}$ is

- $\forall X \in \mathrm{ob}\, \mathcal{C}$, a set of covering sieves $\mathrm{Cov}_J(X)$ satisfying

1) (identity) $\mathrm{Hom}(-,X) \in \mathrm{Cov}_{J}(X)$

(pull-back)

2) $S \in \mathrm{Cov}_{J}(X)$, $f: Y \to X$, then $f^*S \in \mathrm{Cov}_J(Y)$

3) (locality) $S \in \mathrm{Cov}_{\mathcal{C}}(X)$, R sieve on X

If $\forall Y \in \mathcal{C}$, $\forall f \in S(Y)$ $f^*R \in \mathrm{Cov}_{\mathcal{C}}(Y)$

Then $R \in \mathrm{Cov}_{\mathcal{C}}(X)$

- A <u>Grothendieck pre-topology</u> P on $\mathcal{C}$ is

$\forall X \in \mathcal{C}$, a set of covering families

$$\mathrm{Cov}_P(X) = \left(\{X_i \xrightarrow{f_i} X\}_{i \in I} \right)$$

s.t. 1) $Y \to X$ iso $\Rightarrow$ $\{Y \to X\} \in \mathrm{Cov}_P(X)$

2) $\{X_i \to X\} \in \mathrm{Cov}_P(X)$, $Y \to X$, then $X_i \times_X Y$ exists

and $\{X_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}_P(Y)$

3) Composition.

- For a pretop P, the Grothendieck top. J gen. by P.

is $S \in \mathrm{Cov}_{\mathcal{C}}(X) \iff S$ contains some covering family from P

- A site $(\mathcal{C}, T)$

- $F : \mathcal{C}^{op} \longrightarrow Sets$ is a sheaf if F satisfies the sheaf condition w.r.t. all covering sieves

- coverage τ on $\mathcal{C}$

  $X \in \mathcal{C}$, a set of covering sieves $Cov_{\tau}(X)$ satisfying

  $S \in Cov_{\tau}(X)$, $f : Y \longrightarrow X$, then $\exists\, R \subset f^{*} S$, $R \in Cov_{\tau}(Y)$

- (precoverage) Π on $\mathcal{C}$:

  (same as above)

  $\forall \{Y_i \xrightarrow{f_i} X\} \in Cov_{\Pi}(X)$ $\quad Y \xrightarrow{g} X$, then $\exists \{Y_j \xrightarrow{h_j} Y\} \in Cov(Y)$

  $g \circ h_j$ factors through some f_i

$$\begin{tikzcd}
 & \Pi \text{ precoverage} \arrow[dr] & \\
P_{\text{pre-top}} \arrow[ur, dashed] \arrow[dr, "\text{generates}"'] & & \tau \text{ coverage} \arrow[dl, "\text{generates}"] \\
 & T_{\text{top}} &
\end{tikzcd}$$

Site (C, T) If T is generated by a precoverage Π

F presheaf on (C, τ)

is a sheaf $\Longleftrightarrow$ F satisfies the sheaf condition w.r.t. Π

is a sheaf $\Longleftrightarrow$ $\dot{F}$ satisfies the sheaf condition i. II

2. Condensed sets

$$S' \in CH$$

$CH \supset Prof \supset ED$ Gleason $\Longleftrightarrow$ $\forall S' \twoheadrightarrow S$ splits)

↑ compact Hausdorff, ↑ profinite sets, ↑ extremally disconnected

$(U \subset X \text{ open} \Rightarrow \bar{U} \subset X \text{ open})$

profinite sets $\Updownarrow$

compact Hausdorff totally disconnected space

(conn. comp. $= \{x\}$)

Hausdorff + ED $\Rightarrow$ TD.

Def A family $\mathfrak{X} = \{X_i \xrightarrow{f_i} X\}_{i \in I}$ with I finite is called of

type (a) $\coprod X_i \to X$ surj. (finite jointly surj)

(b) $\coprod X_i \to X$ bij

(c) $\mathfrak{X} = \{X' \xrightarrow{p} X\}$, p surj.

Prop $\mathcal{C} \in \{CH, Prof, ED\}$

(a) = pretop on CH, Prof

precov on ED (no fiber products)

(b) + (c) = precov on $\mathcal{C}$

Moreover, (a) & (b)+(c) generate the same topology on $\mathcal{C}$

the condensed top.

D/ (a) $\sim \tau$

(b)+(c) $\sim \tau'$

$\tau' \subset \tau$

$S \in \mathrm{Cov}_\tau(X)$ gen. by $\{X_i \xrightarrow{f_i} X\}_{i\in I}$ type (a)

$f_i : X_i \xrightarrow{\varphi_i} \coprod X_i \xrightarrow{p} X$ $\{\varphi_i\}$ type (b)

$\{p\}$ type c

$S_p \in \mathrm{Cov}_{\tau'}(X)$

(Local character) $\forall Y \in \mathcal{C}$, $f \in S_p(Y)$

Need $f^* S \in \mathrm{Cov}_{\tau'}(Y)$

$\Rightarrow f : Y \xrightarrow{h} \coprod X_i \xrightarrow{p} X$ $f^* S = h^* p^* S$

Suffices: $p^* S \in \mathrm{Cov}_{\tau'}(\coprod X_i)$

follows since $\underset{\text{type (b)}}{\{\varphi_i\}} \in p^* S$. $\square$.

<u>Def</u> $\mathrm{Cond}(\mathrm{Set}) = \mathrm{Sh}(\mathrm{Prof}_{\mathrm{cond}})$

<u>Rk 1</u> $\mathrm{Sh}(\mathrm{CH}) \longrightarrow \mathrm{Sh}(\mathrm{Prof}) \longrightarrow \mathrm{Sh}(\mathrm{ED})$ equivalence

<u>Rk 2</u> Set-theoretic pb.

Prof is large

κ = strong limit cardinal $\quad (\lambda < \kappa \Rightarrow 2^\lambda < \kappa)$

$$\mathrm{Cond}_\kappa(\mathrm{Sets}) := \mathrm{Sh}(\mathrm{Prof}_\kappa)$$

$\kappa < \kappa' \Rightarrow \mathrm{Prof}_\kappa \subset \mathrm{Prof}_{\kappa'} \Rightarrow \mathrm{Cond}_{\kappa'}(\mathrm{Sets}) \to \mathrm{Cond}_\kappa(\mathrm{Sets})$

has a left adj.

$$\mathrm{Cond}_\kappa(\mathrm{Sets}) \longrightarrow \mathrm{Cond}_{\kappa'}(\mathrm{Sets})$$

$$X \longmapsto \Big(T' \longmapsto \operatorname*{colim}_{\substack{T' \to T \\ T\ \kappa\text{-small}}} X(T) \Big)^{\mathrm{Sh}}$$

$$\mathrm{Cond}(\mathrm{Sets}) = \operatorname*{colim}_{\kappa} \mathrm{Cond}_\kappa(\mathrm{Sets})$$

<u>Prop</u> $\mathcal{C} \in \{\mathrm{CH}, \mathrm{Prof}, \mathrm{ED}\}$, $\quad F \in \mathrm{PSh}(\mathcal{C})$

1) $F \in \mathrm{Sh}(\mathrm{CH}/\mathrm{Prof})$

$\Leftrightarrow$ i) $\forall$ finite $\{S_i\}$

$$F(\sqcup S_i) \longrightarrow \prod F(S_i) \quad \text{bijection}$$

ii) $\forall\ S' \twoheadrightarrow S$

$$F(S) \to \{x \in F(S') \mid p_1^*(x) = p_2^*(x) \in F(S' \times_S S')\}$$

bijection

2) $F \in \mathrm{Sh}(\mathrm{ED}) \Leftrightarrow$ i)

Pf/ 1) i) type (b) $S_i \to S = \coprod S_i$ $\quad S_i \times_S S_j = \{\phi \quad i \neq j$

ii) $S' \twoheadrightarrow S$ surj

2) Gleason : $S' \twoheadrightarrow S$ ED $\Rightarrow \exists\ S \to S'$

$\Rightarrow S' \to S$ generates the sieve $\mathrm{Hom}(-, S)$ $\square$.

<u>Example</u> $X \in \mathrm{Top} \rightsquigarrow \underline{X} \in \mathrm{Cond}(\mathrm{Sets})$

$$\underline{X}(S) = C^0(S, X)$$

i) : indiv. proj.

ii) $S' \twoheadrightarrow S \Rightarrow$ quotient

$\{S \to X \text{ cont}\} \leftrightarrow \{S \to S' \to X \text{ cont}\}$.

<u>Ex</u> $\mathbb{R}^{disc} \to \mathbb{R} \in \mathrm{Top}$ bij, not iso

$\Rightarrow \mathrm{Ab}(\mathrm{Top})$ not abelian

But $\mathrm{Cond}(\mathrm{Ab})$ is abelian : $\underline{\mathbb{R}}^{disc} \to \underline{\mathbb{R}}$ has nontrivial cokernel Q

$$Q(S) = \frac{C^0(S, \mathbb{R})}{\{f : S \to \mathbb{R} \text{ loc. cst}\}} \neq 0$$

$$0 \to \underline{\mathbb{R}}^{disc} \to \underline{\mathbb{R}} \to Q \to 0$$

3) Top <u>& Cond(Sets)</u>

Recall: $X \in \mathrm{Top}$ is k-compactly generated if

$\forall\, X \to Y$ is cont. if $\forall\, S \to X$, $S \in CH_k$,

$S \to X \to Y$ is cont.

Prop $\mathrm{Top}^{k\text{-}cg} \subset \mathrm{Top}$ has a right adjoint

$$\mathrm{Top} \longrightarrow \mathrm{Top}^{k\text{-}cg}$$
$$X \longmapsto X^{k\text{-}cg}$$

$X^{k\text{-}cg} = X +$ finest top. s.t. $\forall\, S \to X$, $S \in CH_k$ is continuous

Th 1) $\mathrm{Top} \longrightarrow \mathrm{Cond}_k(\mathrm{Sets})$ has a left adj

$$T \longmapsto \underline{T}$$

$$\mathrm{Cond}_k(\mathrm{Sets}) \longrightarrow \mathrm{Top}$$
$$X \longmapsto X(*)_{top}$$

$X(*)_{top} = X(*) +$ finest top s.t. $\forall\, S \to X(*)$

$S \in CH_k$ coming from a map

$\underline{S} \to X \in \mathrm{Cond}(\mathrm{Sets})$

is continuous

2) $\mathrm{Top} \longrightarrow \mathrm{Cond}_k(\mathrm{Sets})$ faithful

$T \longmapsto \underline{T}$ fully faithful when restricted to $\mathrm{Top}^{k\text{-}cg}$

$\mathcal{D}/2)$: $\mathrm{Top} \longrightarrow \mathrm{Cond}_K(\mathrm{Sets}) \longrightarrow \mathrm{Sets}$ faithful

$T \longmapsto \underline{T} \longmapsto \underline{T}(*) = T$

$\Rightarrow \mathrm{Top} \longrightarrow \mathrm{Cond}_K(\mathrm{Sets})$ faithful

$$\mathrm{Hom}_{\mathrm{Cond}}(\underline{X}, \underline{Y}) \overset{(1)}{=} \mathrm{Hom}_{\mathrm{Top}}(\underline{X}(*)_{top}, Y)$$

$$= \mathrm{Hom}_{\mathrm{Top}}(X^{k\text{-}cg}, Y)$$

$$\overset{X \text{ is } k\text{-}cg}{=} \mathrm{Hom}_{\mathrm{Top}}(X, Y)$$

1) Need: $\forall X \in \mathrm{Cond}_K(\mathrm{Set})$, $\forall Y \in \mathrm{Top}$

$$\mathrm{Hom}_{\mathrm{Top}}(X(*)_{top}, Y) = \mathrm{Hom}_{\mathrm{Cond}}(X, \underline{Y})$$

$X \longrightarrow \underline{Y} \in \mathrm{Cond} \iff X(*) \longrightarrow Y$ s.t.

$\forall S \in CH_K$ comes from $\underline{S} \to X \in \mathrm{Cond}$

the comp. $S \longrightarrow X(*) \longrightarrow Y$ is cont.

$(\Leftarrow)$ $\forall S \in CH$, $X(S) \longrightarrow \underline{Y}(S) = C^0(S, Y)$

$(\underline{S} \to X) \longmapsto S \longrightarrow X(*) \longrightarrow Y$

$(\Rightarrow)$ $X \longrightarrow \underline{Y} \rightsquigarrow X(*) \longrightarrow Y$

$\forall S \in CH_K$ comes from $\underline{S} \longrightarrow X$

$\varphi : \underline{S} \longrightarrow X \longrightarrow \underline{Y}$

$$
\begin{tikzcd}
\underline{S}(S) \arrow[r] \arrow[d] & X(S) \arrow[r] \arrow[d, phantom, "\circlearrowright"] & \underline{Y}(S) \arrow[d] \\
S \arrow[r] & X(*) \arrow[r] & Y
\end{tikzcd}
$$

$$
\underbrace{S \longrightarrow X(*)}_{=\,\eta(S)(\mathrm{id}_S)} \longrightarrow Y
$$

$\square$.