Condensed Mathematics: Motivation

Fangzhou Jin

July 26, 2023

Grothendieck's coherent Duality

- **Coherent duality**: introduced by Grothendieck (1963) as a generalization of Serre duality
  Hartshorne, *Residues and Duality*, 1966

Serre duality	Coherent duality
regular (Cohen-Macaulay) schemes	singular schemes
sheaves dualizing sheaf	complexes of sheaves dualizing complex

- Coherent duality is the prototype for several other duality theories

Other dualities

- Etale duality (SGA4-5)
- Verdier duality (1966), based on Borel-Moore (1960)
- $\mathbb{A}^1$-homotopic duality ($\sim$00's)

Duality	Spaces	Category
coherent duality	Schemes	$D^b_{coh}(Qcoh(X))$
Etale duality	Schemes	$D^b(X_{et}, \Lambda)$
Verdier duality	Topological spaces	$D^b(X, \mathbb{Z})$
$\mathbb{A}^1$-homotopy	Schemes	"Motivic categories"

- The duality theorems can be expressed in terms of the Grothendieck **six functors formalism**
  Six functors: f^*, Rf_*, $Rf_!$, $f^!$, $\otimes^L$, $R\underline{Hom}$
- Similarities as well as major differences between coherent duality and other duality theories

Similarities between dualities

- **Poincaré duality**: for f smooth, $f^!$ differs from f^* by an "orientation sheaf"
- **Atiyah duality**: Rf_* for f smooth proper preserves dualizable objects
- Notion of **dualizing objects** and local (bi)duality
- Uniqueness of the dualizing object up to a $\otimes$-invertible object and a shift
- Dualizing objects are preserved by $f^!$

- **Base change theorems**:
  - Coherent duality: very general base changes (e.g. flat base change)
  - Other dualities: basically only smooth base change
- **Purity**: compare i^* and $i^!$ for i regular closed immersion
  - Coherent duality: **Fundamental local isomorphism**

$$i^*(-) \otimes^L (\det(N_i))^{-1} \simeq i^!(-)$$

  - Other dualities: **Purity transformation** (Déglise-J.-Khan)

$$i^*(-) \otimes^L Th(N_i)^{-1} \to i^!(-)$$

    not an isomorphism in general (see absolute purity)

- In coherent duality, the functor $Rf_!$ does not exist
  Projection formula: for f proper,

$$Rf_* R\underline{Hom}(\mathcal{F}, f^! \mathcal{G}) \simeq R\underline{Hom}(Rf_* \mathcal{F}, \mathcal{G})$$

- In coherent duality, no suitable subcategory of **constructible objects** preserved by 6 functors

Condensed mathematics

Part of these defects are salvaged by the theory of **condensed mathematics** (Clausen-Scholze) or equivalently that of **pyknotic objects** (Barwick-Haine):

- Enlarge the category of sheaves so that $Rf_!$ can be defined too
- Heavy use of (algebraized/categorical version of) analytic geometry
- Ideas close to Grothendieck's work on functional analysis in his early career