§7 proof of the main theorem

Main theorem:

If $f: \mathbb{A}^n_k \to \mathbb{A}^n_k$ has an isolated zero at $\underset{\mathbb{A}^n_k}{\overset{0}{\Uparrow}}$

Then $\deg_0^{\mathbb{A}^1}(f) = \omega_0(f)$ in $GW(k)$.

Lemma 7.1: Let $f: \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a poly. function. Let $y \in \mathbb{A}^n_k(k)$, Suppose $x \in f^{-1}(y)$ is isolated and f is étale at x.

Then $\deg_x^{\mathbb{A}^1}(f) = \omega_x(f)$ in $GW(k)$.

Proof:

$$\begin{tikzcd}
\deg_x^{\mathbb{A}^1}(f) \arrow[rr, equal, "\Uparrow"'] \arrow[dr, equal, "\text{Sun's talk}"'] & & \omega_0(f) \arrow[dl, equal, "\text{Yang's talk}"] \\
& \mathrm{Tr}_{k(x)/k}(\langle J(x)\rangle) &
\end{tikzcd}$$

Proof of our Main thm:

Step 1: We may assume f satisfies Assumption 5.4.

(*) $\exists\, F: \mathbb{P}^n_k \to \mathbb{P}^n_k$ finite flat s.t.

(i) $f = F|_{\mathbb{A}^n_k}$, and $\left(\deg\left(\frac{\mathrm{Frac}\, F_* \mathcal{O}_{\mathbb{P}^n_k}}{\mathrm{Frac}\, \mathcal{O}_{\mathbb{P}^n_k}}\right), \mathrm{char}\, k\right) = 1$.

(ii) $F^{-1}(\underset{0}{\overset{\Downarrow}{\mathbb{A}^n_k}}) \subset \underset{0}{\overset{\Downarrow}{\mathbb{A}^n_k}}$

(iii) F is étale at every point of $F^{-1}(0) \setminus \{0\}$.

In fact, by taking an odd degree extension L/k, when k is a finite field, f satisfies (*).

Since

$$\begin{tikzcd}
GW(k) \arrow[r, "\text{(rank, disc.)}", "\cong"'] \arrow[d, hook, "\omega \mapsto \omega \otimes L"'] & \mathbb{Z} \times k^\times/(k^\times)^2 \arrow[d, hook] \\
GW(L) \arrow[r] & \mathbb{Z} \times L^\times/(L^\times)^2
\end{tikzcd}$$

if $\underline{\mathrm{char}\, k \neq 2}$

(If $\mathrm{char}\, k = 2$, $GW(k) \xrightarrow{\cong} \mathbb{Z}$, the only invariant is the rank)

Step 2: We may assume $\exists\, y_0 \in \mathbb{A}^n_k(k)$ s.t. f is étale at every points of $x \in f^{-1}(y_0)$.

In fact, any field ext'n of degree prime to $\mathrm{char}(k)$ is separable, hence f is étale at the generic point η

Then $\exists$ open nbhd $U \ni \eta$, $f|_U$ is étale.

$Z = \mathbb{A}^n_k - U$, $f(Z) \subset \mathbb{A}^n_k$ is closed

Then any k-rational point of $\mathbb{A}^n_k - f(Z)$ has the desired property. By taking another odd degree field ext'n, we have $\underset{y_0}{\overset{}{(\mathbb{A}^n_k - f(Z))(k)}} \neq \phi$.

Step 3: Compare both sides.

local degree formula.

$$\deg^{\mathbb{A}^1}(f) = \sum_{x \in f^{-1}(y)} \deg_x^{\mathbb{A}^1}(f) \quad \forall y \in \mathbb{A}^n_k(k).$$

$\Rightarrow$ RHS is indep. of the choice of y.

On the other hand,

$\sum_{x \in f^{-1}(y)} \omega_x(f)$ is also indep. of $y \in \mathbb{A}^n_k(k)$

The local terms $\deg_x^{\mathbb{A}^1}(f)$, $\omega_x(f)$ are equal when F is étale at x (Lemma 7.1)

Hence

$$\begin{tikzcd}
\sum_{x \in f^{-1}(y_0)} \deg_x^{\mathbb{A}^1}(f) \arrow[r, equal, "\text{Lemma 7.1}"] \arrow[d, equal, "\leftarrow \text{indep. of } y"] & \sum_{x \in f^{-1}(y_0)} \omega_x(f) \arrow[d, equal, "\leftarrow \text{indep. of } y"] \\
\sum_{x \in f^{-1}(y)} \deg_x^{\mathbb{A}^1}(f) \arrow[r, equal] & \sum_{x \in f^{-1}(y)} \omega_x(f), \ \forall y \in \mathbb{A}^n_k(k)
\end{tikzcd}$$

In particular, $y = 0$.

By (*), F is étale at every point of $F^{-1}(0) \setminus \{0\}$.

Subtracting off the étale terms, we get

$$\deg_0^{\mathbb{A}^1}(f) = \omega_0(f).$$

□

Rmk: In the main theorem, we assume $x = 0 \in \mathbb{A}^n_k(k)$. The same result holds for $x \in \mathbb{A}^n_k$ with $k(x)/k$ separable (Brazelton–Burklund–McKean–Montoro–Opie)

§8 Application: $\mathbb{A}^1$-Milnor number.

classical Milnor number/$\mathbb{C}$

$f \in \mathbb{C}[x_1, \cdots, x_n]$ polynomial

$V(f) = \{x \in \mathbb{C}^n \mid f(x) = 0\}$ hypersurface

$\mathrm{Sing}(f) = \{x \in \mathbb{C}^n \mid f(x) = 0 = \mathrm{grad}(f)(x)\}$

If $p \notin \mathrm{Sing}(f) \overset{\text{implicit function thm}}{\rightsquigarrow} V(f)$ locally near $p \underset{\text{top}}{\sim} \mathbb{C}^{n-1}$

If p is an isolated singularity,

$\mathrm{grad} f: \mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^n_{\mathbb{C}}$

$$\begin{aligned}\mu_p(f) &:= \deg_p^{\mathrm{top}}(\mathrm{grad} f) \\ &= \mathrm{rk}(\omega_p(f)) \\ &= \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \cdots, x_n]_p}{\left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)}\end{aligned}$$

Example 8.1. $f = x^3 - y^2 \in k[x, y]$, $p = (0,0) \in \mathrm{Sing}(f)$ isolated. cusp ≺

$$\mu_0(f) = \dim_{\mathbb{C}}\left(\frac{\mathbb{C}[x,y]_{(x,y)}}{(3x^2, -2y)} \cong \frac{\mathbb{C}[x,y]_{(x,y)}}{(x^2, y)} \cong \frac{\mathbb{C}[x]_{(x)}}{(x^2)}\right)$$

$= 2$

$\mathbb{A}^1$-Milnor number. ($\mathrm{char}\, k \neq 2$)

$f \in k[x_1, \cdots, x_n]$ s.t. $\mathrm{grad}(f): \mathbb{A}^n_k \to \mathbb{A}^n_k$ has an isolated zero at $x \in \mathbb{A}^n_k$.

$$\mu_x^{\mathbb{A}^1}(f) := \deg_x^{\mathbb{A}^1}(\mathrm{grad} f) \in GW(k).$$

Example 8.2 (cusp) $f = x^3 - y^2 \in k[x, y]$. $(0,0)$ cusp.

Assume $\mathrm{char}\, k \neq 2, 3$.

$\mathrm{grad}(f) = (3x^2, -2y)$

$\mathrm{Hess}(f) = \begin{pmatrix} 6x & 0 \\ 0 & -2 \end{pmatrix}$

$\det \mathrm{Hess}(f) = -12x$

$$Q_0(\mathrm{grad} f) = \frac{k[x,y]_{(x,y)}}{(x^2, y)} \cong \frac{k[x]_{(x)}}{(x^2)}$$

with basis $1, x$.

distinguished socle element:

$$3x^2 = 0 + \underset{= a_{11}}{\boxed{3x}}\, x$$

$$-2y = 0 + \underset{= a_{22}}{\boxed{-2}}\, y$$

$$E = \det(a_{ij}) = \det\begin{pmatrix} 3x & 0 \\ 0 & -2 \end{pmatrix} = -6x$$

choose a k-linear map

$$\eta: Q_0(\mathrm{grad} f) \longrightarrow k$$

$$-6x \longmapsto 1$$

i.e. $1 \longmapsto 0$

$x \longmapsto -\frac{1}{6}$

$$Q_0(\mathrm{grad} f) \times Q_0(\mathrm{grad} f) \longrightarrow k$$

$$(a, b) \longmapsto \eta(ab)$$

$$\begin{array}{c|cc} & 1 & x \\ \hline 1 & 0 & -\frac{1}{6} \\ x & -\frac{1}{6} & 0 \end{array}$$

Then EKL-class $\omega_0(\mathrm{grad} f) = \langle -\frac{1}{6} \rangle + \langle -\frac{1}{6} \rangle$

$\cong \langle 1 \rangle + \langle -1 \rangle$

$= h$ hyperbolic form in $GW(k)$.

$\boxed{\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle \text{ if } a \in k^\times}$

α node: simplest singularities.

Defin 8.3. For $\mathrm{char}\, k \neq 2$. Let X be a finite type k-scheme. A point $p \in X$ is called a node if

$$\widehat{\mathcal{O}_{X_{\bar{k}}, \tilde{p}}} \cong \frac{\bar{k}[[x_1, \cdots, x_n]]}{\left(\sum_{i=1}^n x_i^2 + \text{h.o.t.}\right)}$$

for any $\tilde{p} \in X_{\bar{k}}$ lies above p.

Lemma 8.4: Let $f \in k[x_1, \cdots x_n]$. If $f(x) = 0 = \mathrm{grad} f(x)$ and $\mathrm{grad}(f)$ is étale at x. Then

$$x \in \mathrm{Spec}\, \frac{k[x_1, \cdots, x_n]}{(f)} \text{ is a node.}$$

Proof: $k = \bar{k}$, linear transformation assume $x = 0$

$f = \sum a_I x^I$

$f(x) = 0 = (\mathrm{grad} f)(x) \Rightarrow f$ has no constant and linear terms.

change of variables $\rightsquigarrow$

$$f = \sum_{i=1}^n x_i^2 + \text{h.o.t.}$$

i.e. x is a node.

Rmk: If $f = \sum a_i x_i^2$, then $\mathrm{grad}(f) = \langle 2a_1x_1, \cdots, 2a_nx_n \rangle$

$$\begin{aligned}\mu_0^{\mathbb{A}^1}(f) := \deg_0^{\mathbb{A}^1}(\mathrm{grad} f) &= \left\langle \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(0) \right\rangle \\ &= \langle 2^n a_1 \cdots a_n \rangle \\ &= \begin{cases} \langle a_1 \cdots a_n \rangle & \underline{n \text{ even}} \\ \langle 2a_1 \cdots a_n \rangle & n \text{ odd} \end{cases}\end{aligned}$$

In the classical case: $k = \mathbb{C}$:

Milnor shows:

"$\mu_x(f) = \#\{\text{nodes which are bifurcated from the sing. } x\}$"

Example 8.5 $f = x^3 - y^2 \in \mathbb{C}[x, y]$

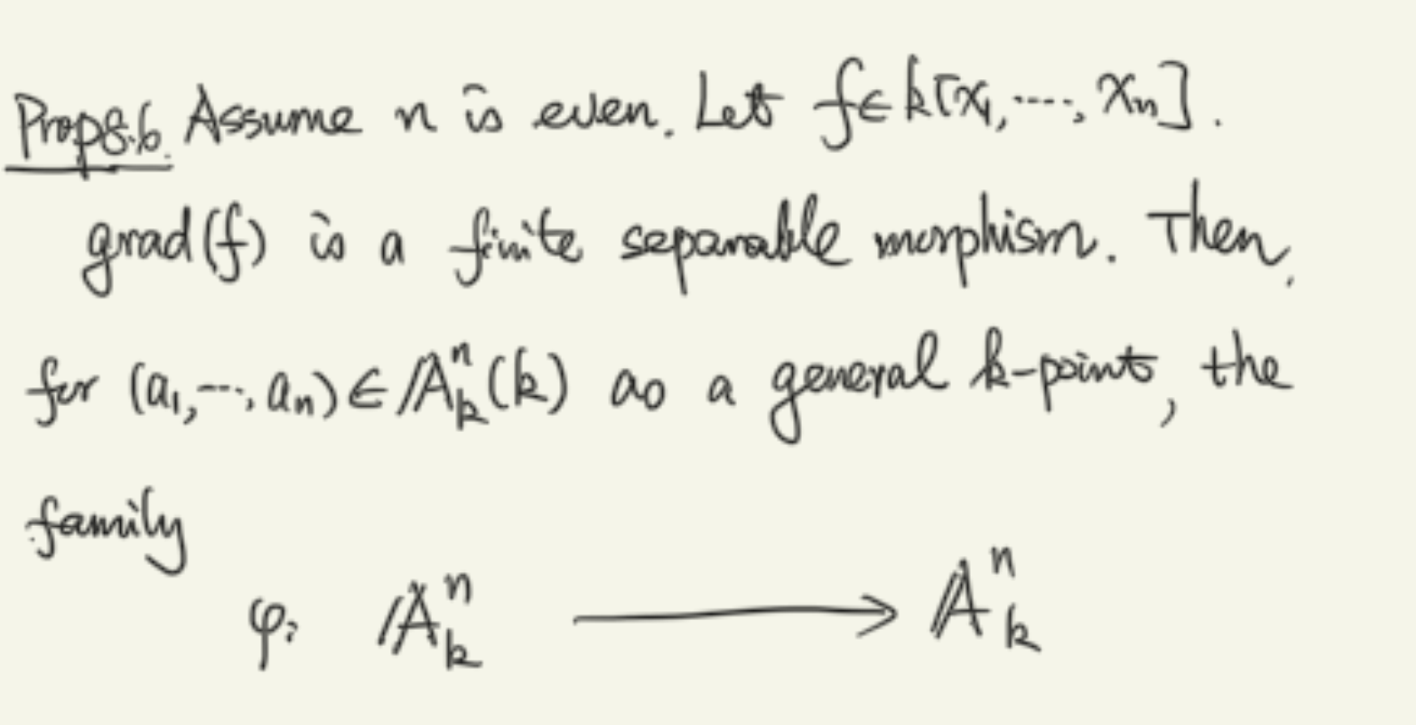
It has an isolated singularity $(0,0)$ (cusp) with the Milnor number $\mu_0(f) = 2$.

Consider:

$$\begin{tikzcd}
Y = \{x^3 - y^2 - ax = t\} \subseteq \mathbb{C}^4 \arrow[d] \\
\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\, \mathbb{C}[t]
\end{tikzcd}$$

Y has singularity $\Leftrightarrow x^3 - ax_1 - t$ has a double roots

$\Leftrightarrow -4a^3 + 27t^2 = 0$



a moves away from 0, the cusp bifurcates into 2 nodes.

$$\text{"}\sum_{x=0} \mu_x(f) = \sum_{y \in \text{nodes}} \mu_y(f - ax)\text{"}$$

The same holds for $\mathbb{A}^1$-theory.

Prop 8.6. Assume n is even. Let $f \in k[x_1, \cdots, x_n]$. $\mathrm{grad}(f)$ is a finite separable morphism. Then, for $(a_1, \cdots, a_n) \in \mathbb{A}^n_k(k)$ as a general k-points, the family

$$\varphi: \mathbb{A}^n_k \longrightarrow \mathbb{A}^n_k$$

$$x \longmapsto f(x) - \sum a_i x_i$$

has only nodal fibers. Suppose that every zero of $\mathrm{grad}(f)$ either has residue field k or is in the étale locus of $\mathrm{grad}(f)$. Then

$$\sum_{x \in \mathrm{Sing}(f)} \mu_x^{\mathbb{A}^1}(x) = \sum_{\substack{\text{Isom. classes} \\ \text{of Henselization}}} \#\left(\begin{matrix}\text{fibers of } \varphi \text{ with} \\ \text{henselization of } \mathrm{Spec}\, \frac{L[x_1, \cdots, x_n]}{(\sum_{i=1}^n u_i x_i^2)}\end{matrix}\right) \cdot \mathrm{Tr}_{L/k}(\langle u_1 \cdots u_n \rangle)$$

in $GW(k)$.

Proof: **Step 1**: $\exists\, \underset{\phi}{\overset{}{U}} \subset \mathbb{A}^n_k$, $U \neq \phi$ open subset, s.t. $\forall (a_1, \cdots a_n) \in U$, the preimage of 0 under

$$\mathrm{grad}(f - \sum a_i x_i): \mathbb{A}^n_k \longrightarrow \mathbb{A}^n_k$$

is étale over k ($\Rightarrow$ they are nodes)

In fact, similar to Step 2 in the proof of the Main Thm.

$V = \{x \in \mathbb{A}^n_k \mid \mathrm{grad}(f) \text{ is étale at } x\} \ni \eta$

$Z = \mathbb{A}^n_k - V$ closed

$U = \mathbb{A}^n_k - f(Z)$ has the desired property.

Step 2: (Lemma 3.9)

Lemma: L/k separable, $x \in X = \mathrm{Spec}\left(L[x_1, \cdots, x_n]/f\right)$, $y \in Y = \mathrm{Spec}\left(k[y_1, \cdots, y_n]/g\right)$ are nodes

If $(X, x) \to (Y, y)$ induces an isom. on Henselizations,

Then $\mu_y^{\mathbb{A}^1}(g) = \mathrm{Tr}_{L/k}\left(\left\langle \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right\rangle\right)$

Proof: We know that

$$\mu_y^{\mathbb{A}^1}(g) = \mathrm{Tr}_{L/k}\left(\left\langle \frac{\partial^2 g}{\partial y_i \partial y_j}(y) \right\rangle\right).$$

$$\det\left(\frac{\partial^2 g}{\partial y_i \partial y_j}(y)\right) \underset{\substack{\uparrow \\ \text{up to a square}}}{\sim} \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)$$

Compute Hess: assume $f = ug$ + n even

Step 3: It is enough to show

$$\sum_{x \in \mathrm{Sing}(f)} \mu_x^{\mathbb{A}^1}(f) = \sum_{x \in \mathrm{Sing}(f - \sum a_i x_i)} \mu_x^{\mathbb{A}^1}(f - \sum a_i x_i).$$

$\| \text{ step 2}$

$$\sum_{\substack{\text{iso. class} \\ \text{of} \\ \text{Hens}}} \#(\text{nods of }) \cdot \mathrm{Tr}_{L/k}\langle u_1 \cdots u_n \rangle.$$

$$\begin{tikzcd}
\mathrm{grad}(f) \text{ étale at } x \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \mathrm{grad}(f - \sum a_i x_i) \text{ étale at } x - a \arrow[d, Rightarrow, "a = (a_1, \cdots, a_n)"] \\
\mu_x^{\mathbb{A}^1}(f) = \mathrm{Tr}_{k(x)/k}\langle \det \mathrm{Hess}(f) \rangle \arrow[r, equal] & \mu_{x-a}^{\mathbb{A}^1}(f - \sum a_i x_i) = \mathrm{Tr}_{k(x)/k}\langle \det \mathrm{Hess}(f - \sum a_i x_i) \rangle
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{grad}(f) \text{ at } x \text{ has residue field } k \arrow[r, Leftrightarrow] \arrow[d, Rightarrow, "\text{Main thm}"'] & \mathrm{grad}(f - \sum a_i x_i) \text{ at } x - a \text{ has residue field } k \arrow[d, Rightarrow] \\
\mu_x^{\mathbb{A}^1}(f) = \omega_x(\mathrm{grad} f) \arrow[r, equal, "\uparrow \text{ linear transform of quadratic forms}"'] & \mu_{x-a}^{\mathbb{A}^1}(f - \sum a_i x_i) = \omega_{x-a}(\mathrm{grad}(f - \sum a_i x_i))
\end{tikzcd}$$

□