2022年1月23日14:00-15:00

Workshop on local A1-Brouwer degree

Lecture 7&8: Some finite determinacy results & Family of symmetric bilinear forms

Recall $\deg : \mathrm{End}_{SH(k)}(1) \longrightarrow GW(k)$.

$f : \mathbb{A}^n_k \longrightarrow \mathbb{A}^n_k$, $x \in f^{-1}(f(x))$ isolated, $f(x)$: k-rational.

Local degree at x	EKL Class at $x \leftarrow 0$
$\deg_x^{\mathbb{A}^1}(f)$ $\quad f_x, f_x'$	$w_x(f) \in GW(k)$, $Q_x(f) \xrightarrow{\phi} k$
	socle $\longmapsto 1$

f étale at x

$$\deg_x^{\mathbb{A}^1}(f) = \mathrm{Tr}_{k(x)/k} \langle J(x) \rangle$$

Same by Galois descent

(小目标)

f: finite

$$\sum_{x \in f^{-1}(\bar{y})} \deg_x^{\mathbb{A}^1}(f)$$

$\bar{y} \in \mathbb{A}_k^n(k)$

independent of $\bar{y}$.

$$\deg^{\mathbb{A}^1}(\bar{f})$$

Same

Harder's thm (小目标)

In general, $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$

$f(0) = 0$

$\boxed{f^{-1}(0) - \{0\}}$

We deform f by $g := f + \underline{\underline{h}}$

h : homogeneous of large degree.

<u>Hope</u> f and g have same Local $\mathbb{A}^1$-degree / EKL Class at $0 \in \mathbb{A}^n_k$, but g is étale at $g^{-1}(0) - \{0\}$, and g come from a base

$$\begin{tikzcd}
\mathbb{A}^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^n \arrow[d] \\
\mathbb{P}^n_k \arrow[r, "\bar{g}"] & \mathbb{P}^n_k
\end{tikzcd}$$

§5 Some finite determinacy results

First Goal f closes to g enough (at 0), $\deg_0^{\mathbb{A}^1}(f) = \deg_0^{\mathbb{A}^1}(g)$, ...

Definition 5.1 $f, g : \mathbb{A}_k^n \to \mathbb{A}_k^n$, $f(0) = g(0) = 0$.

(a) We say f and g are equivalent at the origin ($f \underset{0}{\sim} g$) iff

(1) f and g both have isolated zeros at the origin

(2) $Q_0(f) = Q_0(g)$, $\boxed{E_0(f) = E_0(g)}$

$Q_0(f) = P_{m_0}/(f_1, \dots, f_n)$ $\leadsto$ $\omega_0(f) = \omega_0(g)$

(3) $\deg_0^{\mathbb{A}^1}(f) = \deg_0^{\mathbb{A}^1}(g)$.

(b) $f : \mathbb{A}_k^n \to \mathbb{A}_k^n$ has an isolated zero at the origin.

We say f is finitely determined iff f is b-determined for $b \in \mathbb{N}$,

i.e., $\forall g : \mathbb{A}_k^n \to \mathbb{A}_k^n$, $g_i \equiv f_i \bmod \mathfrak{m}_0^{b+1}$ $(\forall i)$, we have

$f \underset{0}{\sim} g$.

maximal ideal of $\mathcal{O}_x$ at 0
$\parallel$
$k[x_1, \dots, x_n]$

<u>Lemma 5.2</u> Any $f : \mathbb{A}_k^n \to \mathbb{A}_k^n$ with $0 \in f^{-1}(0)$ isolated

is finitely determined (at 0).

proof $Q_0(f) = {}^{P_{m_0}}\!\big/(f_1, \cdots, f_n)$ of finite ~~length~~ length

$\Rightarrow \exists\, b \in \mathbb{N},\ m_0^b \subseteq (f_1, \cdots, f_n)$

Show: $\forall g$ s.t. $g_i \equiv f_i \mod m_0^{b+1} \Rightarrow f \sim_0 g$.

Step 1 show $(f_1, \cdots, f_n) = (g_1, \cdots, g_n)$ $(\Rightarrow \underline{Q_0(f) = Q_0(g)})$

As $g_i \in f_i + m_0^{b+1} \subseteq (f_1, \cdots, f_n) \Rightarrow (g_1, \cdots, g_n) \subseteq (f_1, \cdots, f_n)$.

Show $(f_1, \cdots, f_n) \subseteq (g_1, \cdots, g_n)$

As $g_i \equiv f_i \mod m_0^{b+1}$

$(f_1, \cdots, f_n) \equiv (g_1, \cdots, g_n) + m_0^b \mod m_0^{b+1}$

$$\Rightarrow g_1,\dots,g_n \text{ generate } \frac{(f_1,\dots,f_n)}{m_0^{b+1}} = \frac{(g_1,\dots,g_n)+m_0^{b}}{m_0^{b+1}}$$

$$\Rightarrow \{g_i\} \text{ generate } (g_1,\dots,g_n)+m_0^{b} \Rightarrow m_0^{b} \subseteq (g_1,\dots,g_n).$$

$$f_i \in g_i + m_0^{b+1} \Rightarrow (f_1,\dots,f_n) \subseteq (g_1,\dots,g_n)$$

Step2 $E_0(f) \overset{\text{Socle ele.}}{=} E_0(g)$

Recall $f_i(x) = \overset{0}{\overset{\shortparallel}{f_i(0)}} + \sum_{j=1}^{n} a_{i,j} x_j$

$$\cancel{E(f}\ E_0(f) = \det(a_{i,j}) \in \boxed{Q_0(f)}$$

$$\overset{\shortparallel}{Q_0(g)}$$

$g_i = f_i + \sum b_{ij} x_j \quad , b_{ij} \in m_0^b$

$g_i = \sum (a_{ij} + b_{ij}) x_j$

$\left.\right\} E_0(g) = \det(a_{ij} + b_{ij})$

$\equiv E_0(f) \mod m_0^b$

$\Rightarrow E_0(g) = E_0(f)$ in $Q_0(f)$

Step 3 morphism on Thom spaces

f_x'

$f_0' : \mathbb{P}_k^n / \mathbb{P}_k^{n-1} \cong U / U - \{0\} \xrightarrow{f|_U} \mathbb{A}_k^n / \mathbb{A}_k^n - \{f(0)\} \cong \mathbb{P}_k^n / \mathbb{P}_k^{n-1}$

$= Th(\mathcal{O}_k^n)$

$\deg^{\mathbb{A}^1}(f) = [f_0^!]$ $\in [\ldots]_{SH(k)}(\ldots) \cong GW(k)$

Similar for $g_0^!$.

Show: there is a $\mathbb{A}^1$-homotopy between $f_0^!$ and $g_0^!$.

Write $g_i = \sum n_{ij} f_j$ in P_{m_0}

By def $f_i \equiv g_i \mod m_0^{b+1} \subseteq m_0 \cdot (f_1, \ldots, f_n)$

$\{f_1, \ldots, f_n\}$: basis for $(f_1, \ldots, f_n) / m_0 \cdot (f_1, \ldots, f_n)$

$\Rightarrow (n_{ij}) \equiv id_{n \times n} \mod m_0$

$(n_{ij}) = id_{n \times n} + (m_{ij})$

$\boxed{m_{ij} \in m_0}$

May assume that $m_{ij} \in H^0(V, \mathcal{O})$, $V \subseteq \mathbb{A}^n_k$ Zariski neigh. of 0.

$$H : V \times_k \mathbb{A}^1_k \longrightarrow \mathbb{A}^n_k$$
$$(x, t) \longmapsto M(x,t) \cdot f(x), \qquad M(x,t) = id_{n \times n} + (t \cdot m_{ij}(x))$$

$H^{-1}(0) \supseteq \{0\} \times_k \mathbb{A}^1_k$ Connected component $\qquad 0 \in f^{-1}(0)$

$\|$

$\{0\} \times_k \mathbb{A}^1_k \amalg W$

H induces a map on quotient spaces

$$\frac{V\times\mathbb{A}^1_k}{V\times\mathbb{A}^1_k-\{0\}\times\mathbb{A}^1_k} \hookrightarrow \frac{V\times\mathbb{A}^1_k}{V\times\mathbb{A}^1_k-H^{-1}(0)} \longrightarrow \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k-\{0\}}$$

$$\wr\|$$

$$\frac{V\times\mathbb{A}^1_k}{V\times\mathbb{A}^1_k-\{0\}\times_k\mathbb{A}^1_k} \;\dot\vee\; \frac{V\times\mathbb{A}^1_k}{V\times\mathbb{A}^1_k-W}$$

$$H:\ \frac{V\times\mathbb{A}^1_k}{V\times\mathbb{A}^1_k-\{0\}\times\mathbb{A}^1_k} \longrightarrow \frac{\mathbb{A}^n_k}{\mathbb{A}^n_k-\{0\}} = Th(\{0\}) = \{0\}_+\wedge Th(\mathcal{O}^{\oplus n}_{\mathrm{spec}\,k})$$

$$\{0\} \hookrightarrow \mathbb{A}^n$$

H

$Th(\{0\}\times\mathbb{A}^1_k \hookrightarrow V\times\mathbb{A}^1_k) = (0\times\mathbb{A}^1_k) \wedge Th(\mathcal{O}^{\oplus n}_{\mathrm{Spec}\,k})$

H is a $\mathbb{A}^1$-homotopy from f_0' to g_0'. $\blacksquare$

<u>Prop 5.3</u> $f:\mathbb{A}^n_k\longrightarrow\mathbb{A}^n_k$, $f\neq 0$, $f(0)=0$.

$\exists$ L/k odd degree ext such that $f\otimes_k L$ <u>is equivalent to</u> a function satisfying (Assumption 5.4)

If $|k| = \infty$, then can take $L = k$.

<u>Assump 5.4</u> $f : \mathbb{A}^n_L \longrightarrow \mathbb{A}^n_L$ is <u>a restriction of a</u> morphism $F : \mathbb{P}^n_L \longrightarrow \mathbb{P}^n_L$ such that (1) F finite (<u>flat</u>)

$$\deg\left(\mathrm{Frac}\, F_* \mathcal{O}_{\mathbb{P}^n_L} \Big/ \mathcal{O}_{\mathbb{P}^n_L} \right)$$

coprime to $\mathrm{Char}(L) = p$.

(2) F étale at every point of $F^{-1}(\{0\}) - \{0\}$

(3) $F^{-1}(\mathbb{A}^n_k) \subseteq \mathbb{A}^n_L$.

$f \rightsquigarrow f + \underline{h}$

证明思路 (k infinite)

Choose $d \gg 0$ ($(d,p)=1$, f is d-determined)

H^d_k = affine space of $(h_1, \dots, h_n)$, $h_1, \dots, h_n$ are homoge. poly of degree d

$\mathrm{Good}(H^d_k) = \{ h \in H^d_k \mid \boxed{h^{-1}(0) = \{0\}}$ ✓ okay

$g = f + h$ is étale at every $\}$

point of $g^{-1}(0) - \{0\}$

$\subseteq H_k^d$

$h \in \text{Good}(H_k^d) \subseteq H_k^d$ contains a non-empty Zariski open subset.

$g := f + h \sim_0 f$ $\quad \mathbb{P}_k^n \xrightarrow{F} \mathbb{P}_k^n$ is the required map.

$$[x_0^d : x_0^d f_1(x_1/x_0, \dots, x_n/x_0) + h_1(x_1, \dots, x_n) : \cdots]$$

Lemma 5.5 $\{h \in H_k^d \mid h^{-1}(0) = 0\} \subseteq H_k^d$ is a non-empty

Zariski open subset.

homo. poly $(h_1, \ldots, h_n)$ of deg. d.

<u>proof</u> $(X_1^d, \ldots, X_n^d) \in \{h \in H_k^d \mid \underline{h^{-1}(0) = 0}\} \Rightarrow$ non-empty.

$$I = \mathrm{Image}\left(\left\{ (h,x) \;\middle|\; \begin{array}{l} h_1(x)=0 \\ \vdots \\ h_n(x)=0 \end{array} \right\} \underset{\text{closed}}{\subseteq} \mathbb{P}(H_k^d) \times \mathbb{P}_k^{n-1} \xrightarrow{\text{proper}} \mathbb{P}(H_k^d) \right) \underset{\text{closed}}{\subseteq} \mathbb{P}(H_k^d)$$

$h = [h_1, \ldots, h_n]$

$x = [x_1 : \cdots : x_n]$

$$H_k^d \setminus \{0\} \longrightarrow \mathbb{P}(H_k^d)$$

$$h \in \text{Pre image}(I)^{\text{closed}} \longrightarrow \overset{\cup\!\shortmid}{I}\ \text{closed}$$

$\boxed{H_k^d \setminus \mathrm{Preimage}(I) \cup \{0\}} = \{ h \in H_k^d \mid h^{-1}(0) = 0 \}$ open. $\square$

Lemma 5.6 $f : \mathbb{A}_k^n \to \mathbb{A}_k^n$, $f \neq 0$, $f(0) = 0$.

$(f_i + h_i)(a) = 0$, $i = 1, \dots, n$

$a = (a_1, \dots, a_n) \in \mathbb{A}_k^n(k)$ k-point, $a \neq 0$

assume $\sum \frac{\partial f_i}{\partial x_i}(a) \cdot a_i \neq d \cdot f_i(a)$ for some $d \in \mathbb{N}$, $d \in k^\times$

Then $\left\{ h \in H_k^d \;\middle|\; \begin{array}{l} (f+h)(a) = 0 \\ \det\left(\frac{\partial (f_i + h_i)}{\partial x_j}(a) \right) = 0 \end{array} \right\} \subseteq H_k^d$ Zariski closed of codim $n+1$

proof $f_i(a)+h_i(a)$ $i=1,\dots,n$

$\det\left(\frac{\partial(f_i+h_i)}{\partial x_j}(a)\right)$ regular seq. in H_k^d

$\Rightarrow$

$\Gamma(H_k^d,\mathcal{O})$

Lemma 5.7 $f:\mathbb{A}_k^n\to\mathbb{A}_k^n$, $f\neq 0$, $f(0)=0$

$d\in\mathbb{N}$ s.t. $(d,p)=1$, $d>\max_{1\le i\le n}\{\deg f_i\}$

Then $S_1=\{h\in H_k^d \mid f+h \text{ étale at } (f+h)^{-1}(0)-\{0\}\}\subseteq H_k^d$

contains a <u>non-empty</u> Zariski open subset

<u>proof</u> Show $\dim(\overline{H_k^d \setminus S}) < \dim H_k^d$

$$
\begin{tikzcd}
 & H_k^d \times \mathbb{A}_k^n \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
H_k^d & & \mathbb{A}_k^n
\end{tikzcd}
$$

$S \subseteq H_k^d$

Consider $\Delta = \left\{ (h,a) \in H_k^d \times \mathbb{A}_k^n \;\middle|\; \begin{array}{l} a \in (f+h)^{-1}(0) - 0 \\ \det\left(\frac{\partial(f_i + h_i)}{\partial x_j}(a)\right) = 0 \end{array} \right\}$

$\Delta \subseteq \mathbb{H}_k^d \times \mathbb{A}_k^n \setminus \{0\}$

$\mathbb{H}_k^d \setminus S = \pi_1(\Delta)$

$f \neq 0$, we may assume $f_1 \neq 0$.

$(d,p)=1$, $d > \max\{\deg f_i\} \geq \deg f_1$

$B := \{ a \in \mathbb{A}_k^n \mid \sum \frac{\partial f_1}{\partial x_i}(a) \cdot a_i \neq d \cdot f_1(a) \} \overset{\text{codim } 1}{\subset} \mathbb{A}_k^n$.

bound $\dim \Delta$, $\dim \overline{\pi_1(\Delta)}$.

$\Rightarrow \sum \frac{\partial f_1}{\partial x_i}(x) x_i - d f_1(x) \neq 0$

$(x = (1, 0, \cdots, 0))$

bound $\dim \Delta \cap \pi_2^{-1}(B)$ and $\dim \Delta \cap \pi_2^{-1}(\mathbb{A}_k^n - B)$

Lemma 5.6 $\Rightarrow$ fibers of $\pi_2 : \Delta - \pi_2^{-1}(B) \to \mathbb{A}_k^n - B$ have codim $n+1$ in H_k^d

$\Delta \cap \pi_2^{-1}(\mathbb{A}_k^n - B) \subseteq H_k^d$, $\quad a \in \mathbb{A}_k^n - B$

$$\Rightarrow \dim(\Delta \cap \pi_2^{-1}(\mathbb{A}_k^n - B)) \le \underline{\dim(\mathbb{A}_k^n - B)} + \underline{\dim \Delta_a} < \dim H_k^d$$

$$= n + (\dim H_k^d - n - 1)$$

Similar $\dim \Delta \cap \pi_2^{-1}(B) < \dim H_k^d$

$\Rightarrow \pi_1 : \Delta \longrightarrow H_k^d$ cannot be dominant by dim reasons.

$\Rightarrow H_k^d \setminus \overline{\pi_1(\Delta)}$ are the desired Zariski open subset $\square$

Now prove:

Prop 5.3 $f : \mathbb{A}_k^n \longrightarrow \mathbb{A}_k^n$ $f \neq 0$, $f(0) = 0$.

$\exists\ L|k$ odd degree such that $f \otimes L$ is of the following form:

(1) $f \otimes L = F|_{\mathbb{A}^n_L}$, $F : \mathbb{P}^n_L \to \mathbb{P}^n_L$ finite (flat)

(2) $\operatorname{Frac} F_* \mathcal{O}_{\mathbb{P}^n_L} / \operatorname{Frac} \mathcal{O}_{\mathbb{P}^n_L}$ of degree prime to $p = \operatorname{char}(k)$

(3) F étale at $F^{-1}(0) - \{0\}$

(4) $F^{-1}(\mathbb{A}^n_L) \subseteq \mathbb{A}^n_L$.

If $|k| = \infty$, 取 $L = k$.

<u>proof</u> By Lemma 5.2 $\Rightarrow \exists\, b \in \mathbb{N}$, f is b-determined.

Choose d s.t $(d,p)=1$, $\underline{d>b}$, $d > \max\{\deg f_i\}$

<u>Claim</u> $\exists$ odd degree ext L/k s.t

$\exists\, h \in H_k^d(L)$ s.t $h^{-1}(0)=\{0\}$

s.t $g := f \otimes_k L + h$ étale at every point of $g^{-1}(0) - \{0\}$.

$$\text{Lemma } 5.5 + 5.7 \Rightarrow \left\{ h \in H_k^d(L) \;\middle|\; \begin{array}{l} h^{-1}(0) = \{0\} \\ f \otimes_k L \text{ th étale at } g^{-1}(0) \setminus \{0\} \end{array} \right\}$$

contains a non-empty Zariski open subset $U \subseteq H_k^d$

If k is infinite, then $U(k) \neq \phi \Rightarrow L = k$

If k is finite, $k = \mathbb{F}_q$, $U(\overline{\mathbb{F}_q}) = \bigcup_n U(\mathbb{F}_{q^n})$

$U(\mathbb{F}_{q^n}) \neq 0$. ∎

$g_i := f \otimes L + h$

$\downarrow$

$G : \mathbb{P}^n_L \longrightarrow \mathbb{P}^n_L$ finite, of degree d^n, prime to p.

$(x_0^d : x_0^d f_1(x_1/x_0, \dots) : \dots]$

§6 Family of symmetric bilinear forms

$$f : \mathbb{A}^n_k \longrightarrow \mathbb{A}^n_k \text{ finite.}$$

$$\mathbb{A}^n_k = \operatorname{Spec} P_y$$

Definition 6.1

$$\tilde{Q} := f_* \mathcal{O}_{\mathbb{A}^n_k} = P_x = k[x_1, \ldots, x_n]$$

$$= P_y[x_1, \ldots, x_n] / (y_1 - f_1(x), \ldots, y_n - f_n(x))$$

$\tilde{Q}$ is considered as a P_y-algebra

by

$$\begin{aligned} P_y &\longrightarrow P_x \\ y_1 &\longmapsto f_1(x) \\ &\vdots \\ y_n &\longmapsto f_n(x) \end{aligned}$$

Lemma 6.2 $\tilde{Q}$ is P_y-flat.

$\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n) \in \mathbb{A}^n_k(L)$

$\mathbb{A}^n_k \xrightarrow{f} \mathbb{A}^n_k$

$\underbrace{f^{-1}(\bar{y})}$ $\bar{y}$

L/k finite ext

$$\tilde{Q} \otimes \underline{k(\bar{y})} = \frac{L[x_1, \ldots, x_n]}{(f_1(x) - \bar{y}_1, \ldots, f_n(x) - \bar{y}_n)}$$

$$= \underbrace{Q_{x_1}(f)} \times \cdots \times Q_{x_n}(f).$$

EKL
at x_1

Definition 6.3 (Scheja-Storch)

"An-explicit self duality"

$\tilde{Q}$ finite flat P_y-algebra

($\tilde{Q} = \dfrac{P_y[x_1, \cdots, x_n]}{(y_1 - f_1(x), \cdots, y_n - f_n(x))}$ is of complete intersection over P_y)

By "duality of complete intersection", there is a canonical isomorphism of $\tilde{Q}$-modules

$$\Theta : \mathrm{Hom}_{P_y}(\tilde{Q}, P_y) \xrightarrow{\cong} \tilde{Q}$$

$$\Theta^{-1}(1) \longleftarrow\!\shortmid 1$$

$\underline{\tilde{\eta} := \Theta^{-1}(1)} : \tilde{Q} \longrightarrow P_y$ P_y-linear, "$\underline{\text{generalized trace}}$ $\underline{\text{map}}$"

$\uparrow$ finite P_y-alg.

Symmetric bilinear form

is non-degenerate.

$\mathbb{A}_k^n \xrightarrow{f} \mathbb{A}_k^n$

$x \in f^{-1}(\bar{y}) \rightsquigarrow \bar{y} \in \mathbb{A}_k^n(L)$

$\tilde{\beta} : \tilde{Q} \times \tilde{Q} \longrightarrow P_y$

$\tilde{\beta}(a_1, a_2) = \tilde{\eta}(a_1 \cdot a_2)$

$Q_x(f) \subseteq \tilde{Q} \otimes k(\bar{y})$

$\eta_x := \tilde{\eta}\big|_{Q_x(f)}$

$\beta_x := \beta|_{Q_x(f) \times Q_x(f)}$

$$\boxed{\omega_x = (Q_x(f), \beta_x) \in GW(k)}$$

只取决于 $\eta_x(E_x(f))$

$(\widetilde{Q}, \widetilde{\beta})$ on $\mathbb{A}^n_k$

Lemma 6.4 The distinguished socle element $E = E_0(f)$ satisfies $\eta_0(E) = 1$.

proof $Q_0(f)$-linear homo $\Theta : \mathrm{Hom}_k(Q_0(f), k) \xrightarrow{\simeq} Q_0(f)$

$\eta_0 \longleftrightarrow 1$

(Scheja-Storch)

$$\pi \qquad \longleftrightarrow E$$

$$\begin{array}{ccc} Q_0(f) & \xrightarrow[\varphi]{} & k \\ \downarrow & \text{evaluation} & \\ a & \longmapsto & a(0). \end{array}$$

$\Theta(\pi) = E = E \cdot 1 = E \cdot \Theta(\eta_0) \overset{Q_0(f)\text{-linear}}{=\!=} \Theta(E \cdot \eta_0)$

$$\Rightarrow \boxed{\pi = E \cdot \eta_0} \Rightarrow \pi(1) = (E \cdot \eta_0)(1) = \eta_0(E)$$

$$\| $$

$$1$$

$\blacksquare$

Lemma 6.5 $\tilde{Q}\otimes k(\underline{\bar{y}}) = \underline{Q_{x_1}(f)} \times \cdots \times \underline{Q_{x_m}(f)}$

Then Q_{x_i} is orthogonal to Q_{x_j} w.r.t $\tilde{\beta}\otimes k(y)$.

$i \neq j$

proof $(\tilde{\beta}\otimes k(y))(\underset{\substack{\cap \\ Q_{x_i}(f)}}{a_i}, \underset{\substack{\cap \\ Q_{x_j}(f)}}{a_j}) = (\tilde{\eta}\otimes k(y))(a_i \cdot a_j) = 0$ $\blacksquare$

$[\tilde{Q}\otimes k(\bar{y})] = \sum^{m} [Q_{x_i}(f)]$ in $GW(k)$

$$[Q \otimes R(\bar{y})] = \sum_{i=1} [Q_{x_i}(\bar{y})] \text{ in } GW(k)$$

Lam : Serre's problem on proj module

<u>Lemma 6.6</u> (Harder's theorem :)

Suppose that $(\tilde{Q}, \tilde{\beta})$ is a pair of a finite rank, locally free module $\tilde{Q}$ on $\mathbb{A}^1_k$, and a non-degenerate symmetric bilinear form $\tilde{\beta}$ on $\tilde{Q}$, then : $\forall\ \bar{y}_1, \bar{y}_2 \in \mathbb{A}^1_k(k)$,

$$[(\tilde{Q}, \tilde{\beta})(\bar{y}_1)] = [(\tilde{Q}, \tilde{\beta})(\bar{y}_2)] \text{ in } GW(k)$$

$[(Q, \beta) \otimes k(y_1)] = [(Q, \beta) \otimes k(y_2)]$ in $GW(k)$.

<u>Corollary 6.7</u> $(\tilde{Q}, \tilde{\beta})$ on $\mathbb{A}^n_k$.

$$\boxed{\sum_{x \in f^{-1}(\bar{y})} \omega_x(f) \in GW(k)}$$ is indepent of $\bar{y} \in \mathbb{A}^n_k(k)$.

<u>pf</u> $\displaystyle\sum_{x \in f^{-1}(\bar{y})} \omega_x(f) \overset{\text{Lem 6.5}}{=\!=\!=} [(\tilde{Q} \otimes k(\bar{y}), \tilde{\beta} \otimes k(\bar{y}))]$ in $GW(k$

it is … of $\bar{y}$

H is ind. of g. $\blacksquare$

Harder's thm,

<u>Compute $w_x(f)$ when f is étale at x</u>

<u>Lemma 6.8</u> $f : \mathbb{A}^n_k \longrightarrow \mathbb{A}^n_k$ finite

$\bar{y} \in \mathbb{A}^n_k(k)$, $x \in f^{-1}(\bar{y})$.

If f is étale at x, then

$$w_x(f) = \mathrm{Tr}_{k(x)/k} \quad \langle J(x) \rangle \quad \text{in } GW(k).$$

<u>Note</u> If $k(x)=k$, $\boxed{Q_x(f) = k = \mathcal{P}/\mathfrak{m}.}$

$$J = \det\left(\frac{\partial f_i}{\partial x_j}(0)\right) = \det(a_{ij}) = E$$

$$w_x(f) \quad k \times k \longrightarrow k \qquad \eta : k \longrightarrow k$$

$$\| \qquad \text{okay.} \qquad\qquad E \longmapsto 1$$

$$\langle E \cdot E^{-2} \rangle = \langle E^{-1} \rangle = \langle E \rangle = \langle J \rangle \qquad 1 \longmapsto E^{-1}$$

In general, Use descent for $k(x)/k$

$$x \overset{f}{\longmapsto} y$$

以下设 $k(x) \neq k$

$$\begin{array}{c} L \\ | \\ k(x) \\ | \\ k \end{array} \quad \text{finite Galois ext}$$

$$G = \mathrm{Gal}(L/k).$$

$$S = \left\{ \bar{x} \in \mathbb{A}^n_k(L) \;\middle|\; \begin{array}{c} \bar{x} \text{ is} \\ \text{above} \\ x \end{array} \right\}$$

$$= \left\{ k(x) \hookrightarrow L \right\}$$

$$\begin{tikzcd}
\operatorname{Spec} L \arrow[r, "\bar{x}"] \arrow[d] & \mathbb{A}^n_k \\
\operatorname{Spec} k(x) \arrow[ur, "x"'] &
\end{tikzcd}$$

$V(S) = \operatorname{Hom}(S, L)$ L-algebra

$$V(S) \xrightarrow{V(\phi)} L$$

$$\left(S \xrightarrow{v} L\right) \longmapsto \sum_{s \in S} \frac{1}{J(\bar{x})} v(s)$$

$$\beta_\phi : \quad V(S) \times V(S) \longrightarrow L \circlearrowleft G$$

$$\beta_\phi(v_1, v_2) = V(\phi)(v_1 \cdot v_2) = \sum_{s \in S} \frac{1}{J(\bar{x})} v_1(s) v_2(s).$$

β_ϕ is G-equivariant

$(V(S), \beta_\phi) \circlearrowleft G$

quadratic form $/L$

$\rightsquigarrow$ descent to a quadratic form $/k$

$(Q_x(f), \beta_x) = \mathrm{Tr}_{k(x)/k}\langle J(x)\rangle$ in $GW(k)$.

$\bar{x} \to x$

$Q_x(f)$: polynomial function on $\underline{S}$

$$\downarrow$$

$$\mathrm{Hom}(S, L) = V(S).$$

<u>We show</u>: $\boxed{Q_x(f) \xrightarrow{\cong} \mathrm{Eq}\left(V(S) \rightrightarrows \prod_{\sigma \in G} V(S) \right)}$

$$\mathrm{Eq}(\ldots) = \left\{ S \xrightarrow{v} L \;\middle|\; v(\sigma s) = v(s) \ \forall \sigma \in G \right\}$$

$$E \mapsto 1$$

$$\eta : Q_x(f) \to k$$

$f_x \quad Q_x(f) \to k \qquad V(\phi) \qquad s \in S$

- compatible with bilinear pairing

$$\boxed{V(\phi)\big|_{\substack{Q_x(f) \\ \cap\!\!\mid \\ V(S)}} = \eta_x}$$

x: k-rational

$\rightsquigarrow \quad \eta_x(\underbrace{J_{a,0b}}_{E}) = 1$

$V(\phi)(J_{a,0b}) = 1$

show $(V(S), \beta_\phi)^{\circlearrowleft G}$ deter. $\mathrm{Tr}_{k(x)/k} \langle J(x) \rangle$.

$$\mathrm{Tr}_{k(x)/k} \left\langle \frac{1}{J(x)} \right\rangle$$

$$B : \underline{k(x)} \times k(x) \longrightarrow k$$

$$GW(k)$$

$$B(a, b) = \mathrm{Tr}_{k(x)/k}\left(\frac{ab}{J(x)}\right)$$

$$S = \left\{ k(x) \overset{s}{\hookrightarrow} L \right\}$$

$$\Theta : L \otimes_{\mathbb{R}} \underset{\substack{\uparrow \\ g}}{\mathbb{R}(x)} \xrightarrow{\ \cong\ } \underset{\sim}{V(S)} \qquad L\text{-linear isom}$$

$$(l, g) \longmapsto \left(\Theta(l \otimes g) : S \to L \right)$$

$$\Theta(l \otimes g)(s) = l\, \underline{s(g)}$$

$$\beta_\phi\left(\Theta(1 \otimes g_1), \Theta(1 \otimes g_2)\right) = \sum_{s \in S} \frac{1}{\underset{\substack{\sim \\ \| \\ s(J)}}{J(s)}} s(g_1)\, s(g_2)$$

$$s : \mathbb{R}(x) \longrightarrow L$$

$$g_1 \longmapsto s(g_1)$$

J is defined over k

$$\mathrm{Tr}_{k(x)/k}\left(\frac{g_1 g_2}{J}\right) = B(g_1, g_2) \qquad \blacksquare$$