Morel's A1-degree map

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Recall k field

motivic spaces$/k$ $= PSh(Sm_k, Spaces)$

$\underline{\mathcal{H}_\bullet(k)} = (\text{pointed motivic spaces})[\text{Nis}, \mathbb{A}^1]^{-1}$

Unstable motivic homotopy category

$\mathcal{F} \in \mathcal{H}_\bullet(k) \rightsquigarrow \pi_n^{\mathbb{A}^1}(\mathcal{F})$ = Nis sheaf

associated to

$$U \longmapsto [U \wedge S^n, \mathcal{F}]_{\mathcal{H}_\bullet(k)}$$

$\underline{\mathbb{A}^1\text{-homotopy sheaves}}$

$\underline{SH(k)} = (\mathbb{P}^1\text{-Spectra})$

[stable motivic weak equivalence]$^{-1}$

<u>Goal</u>

<u>Th</u> (Morel) $[\mathbb{1}_k, \mathbb{1}_k]_{\underline{SH(k)}} \cong GW(k)$

$\underline{GW(k)} = K_0$(non-deg. sym. bil. forms over k)

We work <u>unstably</u> in $\mathcal{H}_\bullet(k)$

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3. Main ingredients:

- strongly / strictly $\mathbb{A}^1$-invariant sheaves
- $\mathbb{A}^1$-Hurewicz Theorem
- universality of the (unramified) Milnor-Witt K-theory

Def 1) $\mathcal{F} \in Sh_{Nis}(Sm_k, Grp)$
(sheaf of groups)

$\mathcal{F}$ is strongly $\mathbb{A}^1$-invariant if $\forall X \in Sm/k$

$$H^i_{Nis}(X, \mathcal{F}) \simeq H^i_{Nis}(\mathbb{A}^1_X, \mathcal{F})$$

2) $\mathcal{F} \in Sh_{Nis}(Sm_k, Ab)$

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Ex - homotopy invariant sheaves with transfers are strictly $\mathbb{A}^1$-invariant (Voevodsky)
e.g. Rost's cycle modules.

- $\underline{W}$ unramified Witt sheaf

strictly

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$\underline{I}^j$ powers of the fundamental ideal) strictly $\mathbb{A}^1$-inv.

- $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1_k)$ is strongly $\mathbb{A}^1$-inv

(non abelian).

$\underline{\text{Th}}$ (Morel) $\underline{F} \in Sh_{Nis}(Sm_k, \underline{\underline{Ab}})$

Then F is strictly $\mathbb{A}^1$-invariant

$\Longleftrightarrow$ F is strongly $\mathbb{A}^1$-invariant

- strictly $\Rightarrow$ strongly : since $\underline{Ab} \to \underline{Grp}$ is fully faithful

- strongly $\Rightarrow$ strictly :

$\underline{\text{Th}}$ (Gabber presentation Lemma)

X = localization of a smooth n-dimensional k-scheme at a point

$Y \hookrightarrow X$ closed, codim ≥ 1

Then $\exists$ | $t \in \mathbb{A}^{n-1}_k$, $S = \mathrm{Spec}\, \mathcal{O}_{\mathbb{A}^{n-1}_k, t}$

| $X \to \mathbb{A}^1_S$ étale

s.t. |
$$\begin{tikzcd} & X \arrow[dr] & \\ Y \arrow[ur] \arrow[rr, "\text{closed immersion}"'] & & \mathbb{A}^1_S \end{tikzcd}$$

| Y finite over S

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Rost - Schmid complex

$\mathcal{F} \in Sh_{Nis}(Sm_k, Ab)$ strongly $\mathbb{A}^1$-invariant sheaf

- K field, V = 1-dim^l K-vector space

$$\Rightarrow \quad \mathcal{F}(K, V) := \mathcal{F}(K) \underset{K^\times}{\otimes} V^\times$$

twist

- $\mathbb{G}_m$-loop space : $X \in Sm_k$

$$\mathcal{F}_{-1} : X \longmapsto \mathrm{Ker}\left(\mathcal{F}(\mathbb{G}_m \times X) \longrightarrow \mathcal{F}(X)\right)$$

$$X \xrightarrow{(id, 1)} X \times \mathbb{G}_m$$

is a strongly $\mathbb{A}^1$-invariant sheaf

$$\mathcal{F}_{-n} = (\mathcal{F}_{-(n-1)})_{-1}$$

Def Rost - Schmid complex

$$C^*_{RS}(X; \mathcal{F}) = \bigoplus_{x \in X^{(0)}} \mathcal{F}(\kappa(x), \omega_{X,x}) \longrightarrow \bigoplus_{x \in X^{(1)}} \mathcal{F}_{-1}(\kappa(x), \omega_{X,x}) \longrightarrow \cdots$$

- $\omega_{X,x} = \det(N_x \mathcal{O}_{X,x})$

- boundary = "Norm residue map"

Th 1) C^*_{RS} is a complex

2) $C^*_{RS}(X, \mathcal{F}) \longrightarrow C^*_{RS}(\mathbb{A}^1_X, \mathcal{F})$

2) $C^*_{RS}(X, F) \longrightarrow C^*_{RS}(\mathbb{A}^1_X, F)$

is quasi-isomorphism.

Proof is similar to Rost's arguments for cycle modules.

<u>Th</u> $H^*(\underline{C^*_{RS}(X, F)}) \simeq H^*_{Zar}(X; F) \simeq \underline{H^*_{Nis}(X; F)}$

$\Rightarrow$ F is strictly $\mathbb{A}^1$-inv.

use Gabber + htp. invariance + induction

coniveau spectral sequence.

<u>$\mathbb{A}^1$-Hurwicz Theorem</u>

<u>Def</u> $\mathcal{X}$ = pointed presheaf of spaces, $n \geq 1$

$\mathcal{X}$ is <u>n-connected</u> if $\forall i \leq n$, $\pi_i(\mathcal{X}) = 0$ ← homotopy sheaves

<u>n-$\mathbb{A}^1$-connected</u> if $\forall i \leq n$, $\pi_i^{\mathbb{A}^1}(\mathcal{X}) = 0$ ← $\mathbb{A}^1$-homotopy <u>sheaves</u>.

<u>Th</u> (Morel's unstable $\mathbb{A}^1$-connectivity Theorem)

$n \geq 1$, $\mathcal{X}$ n-connected $\Rightarrow$ $\mathcal{X}$ is n-$\mathbb{A}^1$-connected

<u>Cor</u> $n \geq 2$, $\forall \mathcal{X}$, $\underline{\Sigma^n \mathcal{X}}$ is <u>$(n-1)$-$\mathbb{A}^1$-connected</u>

<u>$\mathbb{A}^1$-homology sheaf</u>

$\mathbb{A}$- homology theory

$\mathcal{X}$ presheaf of spaces $\quad \underline{\pi_n^{\mathbb{A}^1}(\mathcal{X})}$

$\rightsquigarrow \mathbb{Z}(\mathcal{X})$ presheaf of simplicial abelian groups

Dold-Kan

$\rightsquigarrow C_*(\mathbb{Z}(\mathcal{X}))$ normalized chain complex

$\mathbb{A}^1$-localization

$\rightsquigarrow C_*^{\mathbb{A}^1}(\mathcal{X})$ $\quad$ $\mathbb{A}^1$- chain complex.

C_* chain complex $\rightsquigarrow \underline{L_{\mathbb{A}^1}^{(1)}}(C_*)$

$$= \mathrm{cone}\left(\underline{\mathrm{Hom}}(\widetilde{\mathbb{Z}}(\mathbb{A}^1), C_*^f) \xrightarrow{ev_1} C_*^f \right)$$

$$L_{\mathbb{A}^1}^{(1)} \circ L_{\mathbb{A}^1}^{(n-1)}$$

$$L_{\mathbb{A}^1}^{\infty} = \mathrm{colim}\, L_{\mathbb{A}^1}^{(n)}$$

$$\underline{\underline{L_{\mathbb{A}^1}^{ab}(C_*)}} = L_{\mathbb{A}^1}^{\infty}(C_*)^f$$

$\underline{H_n^{\mathbb{A}^1}(\mathcal{X})} = \pi_n^{\mathbb{A}^1}\left(C_*^{\mathbb{A}^1}(\mathcal{X})\right)$ $\quad$ $\underline{\mathbb{A}^1\text{- homology sheaf}}$

$\underline{\text{Th}}$ ($\mathbb{A}^1$- Hurewicz Theorem)

$n \geq 2$ $\quad \mathcal{X}$ $\underline{(n-1)-\mathbb{A}^1\text{- connected}}$

$$\Longrightarrow \underline{\pi_n^{\mathbb{A}^1}}(\mathcal{X}) \xrightarrow{\sim} \underline{H_n^{\mathbb{A}^1}}(\mathcal{X})$$

$\underline{\text{Cor}}$ $\quad n \geq 2 \quad \pi_n^{\mathbb{A}^1}(\underline{\Sigma^n} F) \simeq H_n^{\mathbb{A}^1}(\underline{\Sigma}^n F)$

$$\simeq H_{n-1}^{\mathbb{A}^1}(\Sigma^{n-1}\mathcal{F})$$

$$\simeq \cdots \simeq \widetilde{H}_0^{\mathbb{A}^1}(\mathcal{F})$$

$$\parallel$$

free strictly $\mathbb{A}^1$-invariant sheaf associated to $\mathcal{F}$

Milnor-Witt K-theory

Def F field

$K_*^{MW}(F)$ = graded associative ring with

- generators : $[u]$, $u \in F \setminus \{0\}$, degree $= 1$

  - η, degree $= -1$

- relations :

1) (Steinberg relations) $\forall a \in F \setminus \{0,1\}$

$$[a] \cdot [1-a] = 0$$

2) $\forall a, b \in F \setminus \{0\}$

$$[ab] = [a] + [b] + \eta \cdot [a] \cdot [b]$$

3) $\forall u \in F \setminus \{0\}$, $[u] \cdot \eta = \eta \cdot [u]$

4) $\eta \cdot (\eta \cdot [-1] + 2) = 0$

Rk 1) $K_*^{MW}(F)/\eta \simeq K_*^{M}(F)$ Milnor K-theory

2) $K_0^{MW}(F) \simeq GW(F)$

$$1 + \eta \cdot [a] \longleftarrow\!\shortmid \langle a \rangle^{x \mapsto ax^2}$$

$\eta \cdot [-1] + 2 \longmapsto 1 + \langle -1 \rangle = h$ (hyperbolic form)

3) $\underline{\eta}$ = motivic Hopf map

$\underline{\mathbb{G}_m \wedge \mathbb{P}^1} \simeq \mathbb{A}^2 - \{0\} \longrightarrow \mathbb{P}^1$

$$\downarrow \text{deloop}$$

$\underline{\mathbb{G}_m \to \mathbb{1}}$.

Unramified Milnor-Witt sheaf

$X \in Sm_k$ $\quad \underline{K}_n^{MW}(X) = \operatorname{Ker}\Big(\underline{K_n^{MW}(k(X))} \xrightarrow{\text{norm-residue}} \bigoplus_{x \in X^{(1)}} K_{n-1}^{MW}(k(x)) \Big)$

$\rightsquigarrow \underline{K}_n^{MW} \in Sh_{Nis}(Sm_k, Ab)$ $\underline{\text{unramified MW sheaf}}$

Lemma 1) $\underline{K}_n^{MW}$ is a strongly $\mathbb{A}^1$-inv sheaf

2) $(\underline{K}_n^{MW})_{-1} \simeq \underline{K}_{n-1}^{MW}$

- $\underline{\sigma_n} : (\mathbb{G}_m)^{\wedge n} \longrightarrow \underline{K}_n^{MW}$

$$(u_1, u_2, \dots, u_n) \longmapsto [u_1] \cdot [u_2] \cdots [u_n]$$

Th $\forall$ map of pointed spaces $\phi : (\mathbb{G}_m)^{\wedge n} \to M$

$M \in Sh_{Nis}(Sm_k, Ab)$ strongly $\mathbb{A}^1$-inv.

$\exists ! \, \Phi : \underline{K}_n^{MW} \to M$ s.t. $\Phi \circ \sigma_n = \phi$

$$\begin{tikzcd} (\mathbb{G}_m)^{\wedge n} \arrow[r, "\phi"] \arrow[dr, "\sigma_n"'] & M \\ & \underline{K}_n^{MW}(F) \arrow[u, dotted, "\exists \Phi"'] \end{tikzcd}$$

i.e. σ_n is the universal morphism from $(\mathbb{G}_m)^{\wedge n}$

to strongly $\mathbb{A}^1$-inv. sheaf of abelian groups.

Idea – reduce to fields.

- uniqueness: follows from $K_n^{MW}(F)$ is generated

by $[u_1] \cdots [u_n]$, $u_i \in F \setminus \{0\}$

- Existence: check the relations are satisfied. $\square$

Cor $\underline{K}_n^{MW}$ is the free strictly $\mathbb{A}^1$-inv. sheaf

generated by $(\mathbb{G}_m)^{\wedge n}$.

Cor $n \geq 2$ $i \geq 1$

$$\pi_n^{\mathbb{A}^1}(S^n \wedge (\mathbb{G}_m)^{\wedge i}) \simeq \tilde{H}_0^{\mathbb{A}^1}((\mathbb{G}_m)^{\wedge i})$$

$$\simeq \underline{K}_i^{MW}.$$

Th $\mathcal{X} \in \mathcal{H}_\cdot(k)$ 0-connected

$$[S^n \wedge (\mathbb{G}_m)^{\wedge i}, \mathcal{X}]_{\mathcal{H}_\cdot(k)} \simeq \pi_n^{\mathbb{A}^1}(\mathcal{X})_{-i}(k)$$

Cor $i \geq 1$

$$[(\mathbb{G}_m)^{\wedge j}, (\mathbb{G}_m)^{\wedge i}]_{\mathcal{H}_\cdot(k)} = \underline{K_{i-j}^{MW}(k)}$$

$$\Rightarrow [\mathbb{1}_k, \underbrace{\mathbb{1}_k(n)[n]}_{\mathbb{G}_m^{\wedge n}}]_{SH(k)} = K_n^{MW}(k)$$

$\ldots \underbrace{\ldots}_{\mathbb{G}_m^{\wedge n}}]_{SH(k)} = K_n(k)$

In part, $[\mathbb{1}_k, \mathbb{1}_k] = \underline{GW(k)}$ "0-line"

Rk 1) Th (Röndigs - Spitzweck - Østvær)

$$0 \longrightarrow K^M_{2-n}/24 \longrightarrow \underbrace{\pi_{n+1,n}(\mathbb{1}_k)}_{\text{"1-line"}} \longrightarrow \pi_{n+1,n}\underbrace{f_0(KQ)}$$

$n \geq -4$ exact

effective cover of hermitian K-theory

2) Th (Cazanave)

stable iso classes of non. deg. sym. bil. forms.

$$[\underline{\mathbb{P}^1}, \underline{\mathbb{P}^1}]_{naive} = \boxed{\bigoplus_{n \geq 1} MW^s_n(k) \times k^\times}$$

$\longrightarrow k^\times / k^{\times 2}$ discriminant

- uses Bézout form & resultant.

$\Longrightarrow$ Morel's result is the group completion