Introduction

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- $f : \mathbb{R}^n \to \mathbb{R}^n$ - continuous

- isolated zero at 0

$(f(0)=0, \exists \varepsilon > 0$

$\forall\ 0 < \|x\| < 2\varepsilon,\ f(x) \neq 0)$

$\Rightarrow S^{n-1} \simeq S(0, \varepsilon) \xrightarrow[\|f\|]{f} S^{n-1}$

degree homomorphism: $\simeq [pt, pt]_{sym} = \mathbb{Z}$.

$\deg : [S^{n-1}, S^{n-1}] \longrightarrow \mathbb{Z}$

$\deg_0(f) := \deg\left(\frac{f}{\|f\|}\right) \in \mathbb{Z}$ Local Brouwer degree

- $f \in C^\infty$, $Q_0(f) := C_0^\infty(\mathbb{R}^n)/(f)$ local algebra

$w_0(f) : Q_0(f) \times Q_0(f) \longrightarrow \mathbb{R}$ symmetric bilinear form

(Scheja - Storch form)

Th 1) (Eisenbud - Khimshiashvili - Levine)

$f \in C^\infty \Rightarrow \underline{\deg_0(f)} = \underline{\mathrm{sgn}\,(w_0(f))} \in \mathbb{Z}$

2) (Palamodov)

f real analytic $\Rightarrow$ $f_{\mathbb{C}} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$

$\underline{\deg_0(f_{\mathbb{C}})} = \underline{\mathrm{rk}\,(w_0(f))} \in \mathbb{Z}$

Goal of this workshop

An algebraic version over any field k.

Th (Kass - Wickelgren)

$f : \mathbb{A}^n_k \longrightarrow \mathbb{A}^n_k$ isolated zero at 0

$f(0) = 0$

Then

$\underbrace{\deg_0^{\mathbb{A}^1}(f)} = \underbrace{[w_0(f)]} \in \underbrace{GW(k)}$

$\mathbb{A}^1$-Brouwer local degree (Talk 6)

EKL class

Grothendieck-Witt group of k (Talk 4)

defined using $\mathbb{A}^1$-homotopy theory

$$\mathrm{SymVect}(k) = \{ (V, \rho) \mid V \in \mathrm{Vect}^{fd}(k),\ \rho : V \times V \to k \text{ non degenerate symmetric bilinear form} \}$$

exact category $\quad 0 \to V_1 \to V \to V_2 \to 0$

$$\underline{GW(k)} := \underline{K_0}(\mathrm{Sym\,Vect}(k))$$

Grothendieck group $\quad [V] = [V_1] + [V_2]$

char $k \neq 2$, sym. bil. form. = quad. form.

e.g. - $GW(\mathbb{C}) = \mathbb{Z}$ $\quad$ up to iso, a (nondeg) quadratic form over $\mathbb{C}$ (or any alg. closed field) is uniquely determined by its rank

- $GW(\mathbb{R}) = \mathbb{Z}^2$

over $\mathbb{R}$

q odd - <u>rank</u> & its <u>signature</u>

- $GW(\mathbb{F}_q) = \mathbb{Z} \oplus \mathbb{Z}/2$

$\mathbb{F}_q$

<u>rk</u> & <u>discriminant</u>

$\rightsquigarrow$ quadratic refinement of the classical result,

"arithmetic count" of singularities / enumerative invariants

$\rightsquigarrow$ recovers th. of EKL - Palamodov.

eg.: - $\mathbb{A}^1$- Milnor number (Talk 9)

- (K-W): lines on cubic surface

$$\underline{15\langle 1\rangle + 12\langle -1\rangle}$$

$$\begin{array}{c} 15\{ \\ 12\{ \end{array} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{pmatrix}$$ (instead of 27)

- Conics in $\mathbb{P}^2$ tangent to 5 given conics

$\underline{3264}$ (⟨1⟩ ...

$\frac{\quad}{2}(\langle 1\rangle + \langle -1\rangle)$

(Bachmann – Wickelgren)

– $\mathbb{A}^1$- Euler number (B - W).

. – –

<u>Motivic homotopy theory</u> / <u>$\mathbb{A}^1$- homotopy theory</u>

$I = [0,1] \rightsquigarrow \mathbb{A}^1$

<u>Grothendieck</u> – algebraic K-theory (~1955)
K_0

– étale cohomology (SGA 4 – 5)
1961 – 1963

analogues in AG of invariants from AT.

– <u>standard conjectures</u> / "yoga of motives"

$\rightsquigarrow$ "universal cohomology theory" for algebraic varieties
(<u>smooth</u> & <u>proper</u>)

<u>Quillen</u> : higher algebraic K-theory (197…)

K_i, $i \geq 0$.

Soulé : $Gr_j K_i(X)_{\mathbb{Q}}$:= motivic cohomology with $\mathbb{Q}$-coeff

(~1984)

Bloch : higher Chow groups $CH_i(X, j)$

⇝ first def. of mot. coh. with $\mathbb{Z}$-coeff. (~1985)

Beilinson : motivic coh as analogue of singular cohomology

+ regulators H_B H_{dR} H_ℓ

perverse sheaves (BBDG 1981)

t-structure

triangulated category of mixed motives.

Voevodsky - Thesis (1992)

h-motives : 2 ingredients

- h topology : coverings = universal epimorphisms (EGA)
- $\mathbb{A}^1$ as substitute of $[0, 1]$ htp. inv.

⇝ DM^{eff}

$\mathbb{A}^1$ ||

$$\rightsquigarrow \underline{DM}^{eff}_h(S) = D\Big(Sh_{\underline{\underline{h}}}\big(\underrightarrow{Sch}_S, \underrightarrow{Ab}\big)^{\mathbb{A}^1}\Big)$$

big site

$F(X) = F(\mathbb{A}^1_X)$

- Milnor ($\ell=2$) (1970) / Bloch-Kato (1995) conjecture

k field $\quad \ell \in k^\times$

$$K^M_n(k)/\ell \xrightarrow{\sim} H^n_{et}(k, \mu_\ell^{\otimes n})$$

mod ℓ $\underline{\text{Milnor K-theory}}$ $\qquad$ Galois cohomology

(Now a theorem of Voevodsky, Suslin & Rost)

"$\underline{\text{Norm - Residue Theorem}}$"

$\underline{\text{Idea}}$: Study "extraordinary cohomology theories" in AG.

$\rightsquigarrow$ generalise the proof of B-K conj in weight 3 by Merkurjev - Suslin

Voevodsky (~2000) : motivic cohomology

& triangulated category of mixed motives

(also: Hanamura / Levine)

- new ingredients : - finite correspondences & transfers.

- Nisnevich / étale topology

X_{Nis} = Nisnevich site

Cat : $Y \to X$ étale

Covering : $(Y_i \to X)_i$ covering

$\Longleftrightarrow$ 1) surjective.

2) $\forall x \in X, \exists i, \exists y \in Y_i$

$k(x) \xrightarrow{\sim} k(y)$

$$DM^{eff}_{(Nis)}(k) = D\left(\underline{Sh}^{tr}_{Nis}(Sm_S, Ab)^{\mathbb{A}^1}\right)$$

smooth S-schemes

homotopy invariant Nis sheaves with transfers.

$a_{et} \downarrow$

$a_{et} \Big\downarrow$ $\qquad \Big|$ smooth S-schem

homotopy invariant Nis sheaves with transfers.

$DM^{eff}_{et}(k)$ = same thing with etale topology

<u>Th</u> (Suslin - Voevodsky rigidity) $n \in k^\times$

$$DM^{eff}_{et}(k, \mathbb{Z}/n) \xrightarrow{\sim} D(Sh(k_{et}, \mathbb{Z}/n))$$

$X \in Sm/k$

$$\begin{array}{ccc}
[(\mathbb{Z}/n)^{tr}(X), \mathbb{Z}/n(q)[p]]_{DM^{eff}_{Nis}} & \longrightarrow & [(\mathbb{Z}/n)^{tr}_{et}(X), (\mathbb{Z}/n)_{et}(q)[p]]_{DM^{eff}_{et}} \\
\wr\wr & & \wr\wr \\
H^{p,q}_M(X, \mathbb{Z}/n) & \longrightarrow & H^p_{et}(X, \mu_n^{\otimes q})
\end{array}$$

$p = q$, $X = \operatorname{spec} k$

$$K^M_p(k)/n \xrightarrow{\sim} H^p_{et}(k, \mu_n^{\otimes p})$$

(Suslin - Voevodsky, Rost 2002-2005)

Morel - Voevodsky (2000) : $\mathbb{A}^1$-homotopy theory

$$\begin{tikzcd} \mathcal{H}_\bullet \arrow[d] \\ SH \arrow[r, shift right, "N"'] \arrow[d, equal] & D(Ab) \arrow[l, shift right, "K"'] \\ \mathcal{H}_\bullet[(S^1)^{\wedge}]^{-1} \end{tikzcd}$$

$\mathcal{H}_\bullet$ = category of topological spaces up to homotopy

stable Dold-Kan correspondence.

$\mathcal{H}_\bullet[(S^1)^{\wedge}]^{-1}$ stable homotopy category

objects: S^1-spectra

$$E = (E_n)_{n \in \mathbb{N}} \qquad E_n \in \text{Spaces}.$$

$$+ \ \sigma_n : S^1 \wedge E_n \longrightarrow E_{n+1} \qquad \text{suspension maps}$$

morphism: $E \xrightarrow{f} F = (E_n \xrightarrow{f_n} F_n)$ commutes with σ_n.

continuous

stable homotopy groups:

$$\pi_n(E) = \operatorname*{colim}_i \pi_{n+i}(E_i)$$

Ex $X \in Top_\bullet$, $E_i = X \wedge S^i$

(i.e. $E = \Sigma^\infty X$ infinite suspension)

Then for $i > n$, the sequence $i \mapsto \pi_{n+i}(E_i)$ is independent of i

(Freudental suspension theorem)

- $f: E \to F$ is a stable weak equivalence if it induces iso on stable homotopy groups.

$$SH_{top} = (S^1\text{-spectra})[\text{s.w.e.}]^{-1}$$

- SH is a triangulated category, shift $=$ S^1-suspension.

- Every object represents a cohomology theory

$$E^n(X) = [X, E \wedge S^n]_{SH_{top}}.$$

eg - suspension spectra $X \in Top_\bullet$

$\Sigma^\infty X \in SH$.

In part, <u>sphere spectrum</u> $S = \Sigma^\infty(pt)$.

- <u>Eilenberg-MacLane spectra</u>: A com. ring.

$HA = (K(A,1), K(A,2), \cdots)$

EM spaces.

represents singular coh. with coeff in A.

- MU: complex cobordism

tmf: top. modular form

...

- From ∞-categorical point of view,

SH = Stabilization of Spaces

= universal stable (Δ)-category.

<u>Motivic homotopy</u>

<u>Def</u> S = scheme

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Spaces = category of spaces (eg. simplicial sets or CW-complexes)

A <u>motivic space</u> over S is a presheaf of Spaces over $\underbrace{Sm}_{\text{cat. of smooth } S\text{-schemes}}{}_S$

$\mathcal{H}(S) = (\text{motivic spaces}/S)\,[\text{Nis}, \mathbb{A}^1]^{-1}$

localize w.r.t. Nis. top. & projections $Y \times \mathbb{A}^1 \to Y$

<u>Unstable motivic homotopy category</u>

$\mathcal{H}_\bullet(S)$ = pointed sheaves ...

<u>homotopy sheaves</u> $X \in \mathcal{H}_\bullet(S)$

$\pi^{\mathbb{A}^1}_{a,b}(X)$ = Nis. sheaf on Sm_S associated to

$$\mathcal{U} \longmapsto [U \wedge S^{a-b} \wedge \mathbb{G}_m^{b}, X]_{\mathcal{H}_\bullet(S)}$$

<u>Stabilization</u>.

Stabilization

Def A $\mathbb{P}^1$-spectrum (motivic spectrum) is $E = (E_n)_{n\in\mathbb{N}}$ $E_n \in \mathcal{H}_\bullet(S)$

+ suspension $\sigma_n : \mathbb{P}^1 \wedge E_n \to E_{n+1}$

stable motivic weak equivalence

$f: E \to F$ induces iso on htp. sheaves.

$$SH(S) = (\mathbb{P}^1\text{-Spectra})[\text{s.m. w.e.}]^{-1}$$

stable motivic homotopy category

two spheres : $\underset{\underset{\mathbb{1}(1)[2]}{\|}}{\mathbb{P}^1} \underset{\mathbb{A}^1}{\sim} \underset{\underset{\mathbb{1}[1]}{\|}}{S^1} \wedge \underset{\underset{\mathbb{1}(1)[1]}{\|}}{\mathbb{G}_m}$

- $SH(S)$ is triangulated by S^1-suspension

- $SH(S)$ is the universal stable ∞-cat which satisfies Nisnevich descent & $\mathbb{A}^1$-invariance (Robalo, Drew-Gallauer)

$\boxed{\text{Nis sheaves of spaces}}$

$\boxed{\text{Nis sheaves of } \underline{\text{spaces}} \text{ on } Sm_S}$

Drew-Gallauer)

$\boxed{\text{Nis sheaves of } \underline{Ab.} \text{ with } \underline{\text{transfers}} \text{ on } Sm_S}$

$$\begin{tikzcd} \underline{\mathcal{H}_\bullet(S)} \arrow[r, shift left] \arrow[d, "\Sigma^\infty"'] & \underline{DM^{eff}(S)} \arrow[l, shift left] \arrow[d] \\ \underline{SH(S)} \arrow[r, shift left] & \underline{DM(S)} \arrow[l, shift left] \end{tikzcd}$$

motivic Dold-Kan correspondence

- Every object in $SH(S)$ represents a bigraded cohomology theory

$$E^{p,q}(U) = [U, (S^1)^{\wedge(p-q)} \wedge (\mathbb{G}_m)^{\wedge q} \wedge E]_{SH(S)}$$

eg. - $H\mathbb{Z}$ <u>motivic E-M spectrum</u>, represents <u>motivic cohomology</u>.

Th (Cisinski-Déglise) S regular, $DM(S) =$ modules over $H\mathbb{Z}$ $\subset SH(S)$

- $KGL = \mathbb{Z} \times \underline{BGL_\infty}$ represents homotopy K-theory (Morel-Voevodsky, Riou)

- MGL represents algebraic cobordism (Levine - Morel)

Rationally

$$\begin{tikzcd} SH(S)_{\mathbb{Q}} \arrow[r, two heads, shift left] & DM(S)_{\mathbb{Q}} \arrow[l, shift left] \arrow[r, "\sim"] & DM_{et}(S)_{\mathbb{Q}} \end{tikzcd}.$$

split epi

Six functors in SH

- originates from Grothendieck's theory for l-adic sheaves (SGA 4)
- developed in $\mathbb{A}^1$-homotopy by Voevodsky ("cross functors", unpublished), Ayoub, Cisinski - Déglise

- $f : X \to Y \in Sch$ separated of finite type

$$f^* : SH(Y) \rightleftarrows SH(X) : f_*$$

$f_! : SH(X) \rightleftarrows SH(Y) : f^!$

exceptional direct image

exceptional inverse image.

$(\otimes, \underline{Hom})$ closed symmetric monoidal

$\parallel$

$\wedge$ Smash product

$f : X \to Y$

<u>Constructions</u> - f^* is pullback on schemes

$$(f^* F)(W) = \operatorname*{colim}_{\substack{U \in Sm_Y \\ W \to U \times_Y X}} F(U)$$

- $f_* (= Rf_*)$ = (derived) direct image on schemes

= right adjoint of f^*.

When f $\boxed{\text{smooth}}$, f^* exact $\Rightarrow$ has left adjoint

$\underline{\underline{f_\#}} : SH(X) \rightleftarrows SH(Y) : \underline{f^*}$

<u>Th</u> (Nagata compactification)

$f : X \to Y$ separated of finite type

X, Y qc qs

X, Y qc qs

$$\Rightarrow \exists \quad \begin{tikzcd} & P \arrow[dr, "\bar f"] & \\ X \arrow[ur, "j"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

$\bar f$ proper

j open immersion.

<u>Def</u> (Deligne's method)

$$f_! := \bar f_* \, j_\# : SH(X) \longrightarrow SH(Y)$$

- $f^!$ = right adjoint of $f_!$

<u>Check</u> - $f_!$ indep of choice

- is compatible with compositions.

<u>Properties</u> given by <u>axioms</u>. SeSch.

<u>(homotopy)</u> : $p : \mathbb{A}^1_S \to S$

$p^* : SH(S) \longrightarrow SH(\mathbb{A}^1_S)$ is fully faithful

$(\Leftrightarrow 1 \xrightarrow{\sim} p_* p^*)$

<u>(Localization)</u> : $Z \overset{i}{\hookrightarrow} S$ closed immersion

(see BBDG) $U \overset{j}{\hookrightarrow} S$ open complement.

Thm 1) The pair of functors

$$(i^*, j^*): SH(S) \longrightarrow SH(Z) \times SH(U)$$

is conservative. $A \longmapsto 0 \implies A = 0$.

2) $i^* i_* \simeq 1$. $j^* i_* = 0$.

$\implies j_! j^* \to 1 \to i_* i^* \xrightarrow{+1}$ cofiber seq.

$i_* i^! \to 1 \to j_* j^* \xrightarrow{+1}$ (= distinguished triangles)

Def $f: X \to S$ smth. Define

$$M_S(X) = f_\# \mathbb{1}_X \in SH(S)$$

sphere. $= f_! f^! \mathbb{1}_S \neq f_! \mathbb{1}_X$ Borel-Moore motive.

"(homological) motive of X over S"

or $Y \xrightarrow{\text{sm}} X \xrightarrow{\text{sm}} S$ viewed as a smooth space over S

$\boxed{\mathbb{P}^n_k / \mathbb{P}^{n-1}_k}$

- $Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U = X - Z$, 闭 i, $\pi: X \to S$

quotient / cofiber

$$\begin{array}{ccccc} M_S(U) & \longrightarrow & M_S(X) & \longrightarrow & M_S(X/X-Z) \\ \| & & \| & & \| \\ f_\# j_! j^* \mathbb{1}_X & \longrightarrow & f_\# \mathbb{1}_X & \longrightarrow & f_\# i_* i^* \mathbb{1}_X \\ & & & & \| \\ & & & & f_\# i_* \mathbb{1}_Z \end{array}$$

$E \in SH(X)$

In particular $\rightsquigarrow$ ...

$E \in \mathcal{SH}(X)$

In particular

$f: X \to S$ smooth

$s: S \to X$ section

$\underline{E_n(X/S)} = [\mathbb{1}_X, E \wedge f^! \mathbb{1}_S]$ $\mathcal{SH}(X)$

$f: X \to S$ etale.

BM-theory.

$E = H_{et} \mathbb{1}$

$$Th(f,s) := f_\# s_* : \mathcal{SH}(S) \to \mathcal{SH}(S)$$

<u>Thom transformation</u>

(<u>Thom stability</u>) : $\forall f: X \to S$ smooth

$s: S \to X$ section

$Th(f,s)$ is an equivalence.

<u>Ex</u> $p: V \to S$ vector bundle

$s: S \to V$ zero section

$$Th_S(V) = p_\# s_* \mathbb{1}_S \in \mathcal{SH}(S)$$

<u>Thom space</u> of V

$$V = \mathbb{A}^n_S \Rightarrow Th_S(\mathbb{A}^n_S) \cong \mathbb{1}_S(n)[2n].$$

DM 中，$Th(V) \simeq \mathbb{1}(d)[2d]$

SH 中不行。

The 3 axioms (homotopy), (Localization) & (Thom stability) will imply all other properties

Check for SH: (homotopy) is by def.

(Localization): Morel - Voevodsky

(Thom stability): reduce to $\mathbb{A}^1_S \to S$

then check directly (Ayoub)

(at the level of model categories)

Morel - Voevodsky purity

$f: X \to S$ smooth

$$\begin{array}{ccc} X \times_S X & \xrightarrow{p_2} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & S \end{array}$$

$$X \xrightarrow{\delta} X \times_S X$$

Def $\Sigma_f = Th(p_1, \delta) = p_{1\#} \delta_*$

$$SH(X) \longrightarrow SH(X)$$

$\Rightarrow \mathfrak{p}_f : \underline{f_\#} = f_\# p_{2*} \delta_* \longrightarrow f_* p_{1\#} \delta_* = \underline{f_* \Sigma_f}$

$\underline{(*_f)}$; – Σ_f is an equivalence

– $\mathfrak{p}_f : f_\# \longrightarrow f_* \Sigma_f$ is iso.

Th (Voevodsky - Röndigs - Ayoub)

$\forall f$ smooth

(htp) + (Loc) + (Thom st.) $\Rightarrow$ $(*_f)$

Idea – first prove $(*_{\mathbb{P}^n})$

– Then use Chow lemma & induction. $\square$

Let $\begin{tikzcd} Z \arrow[r, "\text{cl. imm.}"] \arrow[dr, "\text{sm}"'] & X \arrow[d, "\text{sm}"] \\ & S \end{tikzcd}$.

Let $D_Z X = Bl_Z(\mathbb{A}^1_X) - Bl_Z(X)$

deformation to the normal cone

$$\begin{tikzcd} N_Z X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D_Z X \arrow[d] & X \times \mathbb{G}_m \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner"] \\ 0 \arrow[r] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l] \end{tikzcd}$$

Th (M-V):

$$M_S(X/X-Z) \xrightarrow{\sim} M_S(D_Z X / D_Z X - \mathbb{A}^1_Z) \xleftarrow{\sim} M_S(N_S X / N_S X - 0)$$

$$\parallel$$

$$Th_S(N_Z X)$$

<u>Idea</u> : Nisnevich locally,

(X, Z) is étale over $(\mathbb{A}^{n+c}_S, \mathbb{A}^n_S)$

$\rightsquigarrow$ reduce to the case $X = \mathbb{A}^c_Z$, then use (htp). $\square$.

<u>Cor</u> $f : X \to S$ smooth $\quad K \in SH(X)$.

$$\begin{array}{ccc} \Sigma_f(K) & \simeq & Th_X(N_\Delta(X \times_S X)) \otimes K \\ \parallel & & \wr\wr \\ X \times_S X / X \times_S X - \Delta & & T_f \end{array}$$

$$\simeq Th_X(T_f) \otimes K$$

$$\Rightarrow f_\#(K) \overset{(*_f)}{\simeq} f_!(\Sigma_f K) \simeq f_!(Th_X(T_f) \otimes K)$$

By adjunction, we get

f smooth $\Longrightarrow$ $\underline{f^!}K \simeq \underline{f^*}K \otimes \underline{Th_X(T_f)}$

(<u>relative purity</u>)

<u>Th</u> (6 functors)

$f: X \to Y$ (f^*, f_*)

sep. F.T. $(f_!, f^!)$

$(\otimes, \underline{Hom})$

1) f^* is symmetric monoidal

2) $\exists$ natural transformation $f_! \longrightarrow f_*$

isomorphism if f is proper.

3) f smooth $\Longrightarrow$ $f^! \simeq f^* \otimes Th(T_f)$

4)
$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$p^* f_! = g_! q^*$

$q_* g^! \simeq f^! p_*$

5) $Z \overset{i}{\hookrightarrow} S \overset{j}{\hookleftarrow} U$ $j_! j^! \longrightarrow 1 \longrightarrow i_* i^*$

5) $Z \xhookrightarrow{i} S \xhookleftarrow{j} U$ $\qquad j_! j^! \to 1 \to i_* i^*$

cofiber sequence

6) $(f_! K) \otimes L \simeq f_!(K \otimes f^* L)$

$\underline{Hom}(f_! K, L) \simeq f_* \underline{Hom}(K, f^! L)$

$f^! \underline{Hom}(L, M) \simeq \underline{Hom}(f^* L, f^! M)$

<u>Fundamental class</u>

- $X \in Sch$, $0 \to E_1 \to E \to E_2 \to 0 \in Vect_X$

ses of vector bundles

$\Rightarrow Th(E) \simeq Th(E_1) \otimes Th(E_2)$

$\Rightarrow$ Thom space extends to a map

$\otimes$-invertible

$$Th : K_0(X) = K_0(Vect_X) \longrightarrow SH(X)^{\otimes}$$

<u>Def</u> $f: X \to Y$ is a <u>local complete intersection</u> (lci)

if $\exists$
$$\begin{tikzcd} & P \arrow[dr, "g"] & \\ X \arrow[ur, "i"] \arrow[rr, "f"'] & & Y \end{tikzcd}$$
i regular closed immersion

$X \xrightarrow{f} Y$ $\quad$ i closed immersion, g smooth

$\rightsquigarrow$ $\tau_f := -[N_i] + i^*[T_g] \in K_0(X)$

<u>virtual tangent bundle</u>

<u>Th</u> (Déglise - J. - Khan)

$f: X \to Y$ lci $\Rightarrow$ $\exists$ natural transformation

$$\underline{f^* \otimes \mathrm{Th}(\tau_f) \longrightarrow f^!}$$

(<u>purity transformation</u>)

- compatible with composition
- for f smooth, agrees with relative purity

<u>Idea</u> - define fundamental classes

- smooth case is relative purity
- regular closed immersion: use deformation to the normal cone + a variant of Fulton's construction.

$\Rightarrow$ Establish intersection theory in SH.